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Abstract
The quantum periodic XXZ chain with alternating spins is studied. The properties of the related R-matrix and Hamiltonians are discussed. A compact expression for the ground-state energy is obtained. The corresponding conformal anomaly is found via the finite-size computations and also by means of the Bethe ansatz method. In the presence of an external magnetic field, the magnetic susceptibility is derived. The results are also generalized to the case of a chain containing l different spins.

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1. Introduction

$(1 + 1)$-dimensional integrable, exactly solvable, field theories and one-dimensional integrable lattice models have been extensively studied during the last two decades. They found numerous applications in a variety of areas in statistical physics [1, 2], condensed matter physics [3] and high energy physics [4]. Apart from that, integrable models have their own mathematical interest in the context of quantum groups [5, 6].

The best known examples of quantum integrable models are the sine-Gordon model for which physical quantities such as spectrum and exact scattering matrices have been determined [7], and the Heisenberg model solved originally by Bethe [8]. For the Heisenberg model as well, quantities of physical interest, such as spectrum, scattering matrices and correlation functions, have been explicitly derived [2, 9, 10].

An integrable generalization of the Heisenberg model to higher spins and its q-deformation (the XXZ model) have been extensively studied in the literature (see, e.g., [11–18]). In this paper we will investigate the integrable alternating XXZ spin chain, i.e., an inhomogeneous chain with spin S_1 at odd sites and spin S_2 at even sites. The first study of such a model, with

$(S_1, S_2) = \left(\frac{1}{2}, 1\right)$, was done in [19]. Later, various aspects of alternating chains were also considered in [20–25] but mostly for specific cases such as $\left(\frac{1}{2}, 1\right)$, $\left(\frac{1}{2}, S\right)$ and $\left(S, S + \frac{1}{2}\right)$. Our aim is to elaborate the generic (S_1, S_2) case in detail in the framework of the algebraic Bethe ansatz with a particular emphasis on the thermodynamic limit and the finite-size correction analysis and then to generalize the results to the case of l spins. In the present paper we will consider only the case of periodic boundary conditions. Some related results on open chains were presented in [26].

The paper is organized as follows. In section 2 we outline the basic ingredients of the algebraic Bethe ansatz approach for the model in question; in particular, we discuss the structure and properties of the related R-matrix and Hamiltonians (technical details are given in appendix A). In section 3 we consider the model at zero temperature. We show that its ground-state energy can be expressed in terms of a specific q-deformation of the Ψ-function (related to the Barnes double gamma function). Here we also present some finite-size computations of the corresponding central charge. In section 4 we first derive the Bethe equations in the thermodynamic limit at finite temperature T and in the presence of a magnetic field H. Then we compute the resulting magnetic susceptibility in the $T \to 0$ limit. We also derive the effective conformal anomaly of the system by means of the thermodynamic limit of the algebraic Bethe ansatz and remark that the underlying conformal theory should have a spinon basis related to the D_n algebra. In section 5 we generalize our results to the case of the XXZ chain containing l different spins.

For compactness of notation, we use in the text both the deformation parameter γ and $q \equiv e^{i\gamma}$. In section 2 we assume that q is either real or takes values on the unit circle, whereas the results of sections 3, 4 and 5 are valid only for the latter case.

2. Algebraic Bethe ansatz

2.1. R-matrix

The underlying algebra of the XXZ model is the quantum Lie algebra $U_q(sl_2)$ with the following defining relations [11],

$$[S^+, S^-] = [2S^3]_q \qquad [S^3, S^\pm] = \pm S^\pm \tag{1}$$

where

$$[t]_q = \frac{q^t - q^{-t}}{q - q^{-1}} \tag{2}$$

is the standard definition of q-number. For generic q algebra (1) has the same structure of representations as the undeformed algebra sl_2. In particular, irreducible highest weight representations V_S are parametrized by non-negative integer of half-integer spin S and are $(2S + 1)$-dimensional. In this case, the generators (1) regarded as elements of $\mathrm{End}(\mathbb{C}^{2S+1})$ admit the following matrix realization,

$$(S^+)_{m,n} = (S^-)_{n,m} = \sqrt{[m]_q[2S+1-m]_q}\,\delta_{m+1,n} \qquad (S^3)_{m,n} = (S+1-m)\delta_{m,n} \tag{3}$$

where $m, n = 1, \ldots, 2S + 1$.

An L-operator associated with the XXZ model can be chosen as follows:

$$L^S(\lambda) = \begin{pmatrix} [S^3 - i\lambda]_q & q^{-(\frac{1}{2}+i\lambda)} S^- \\ q^{\frac{1}{2}+i\lambda} S^+ & -[S^3 + i\lambda]_q \end{pmatrix}. \tag{4}$$

The defining relations (1) are then equivalent to the following matrix exchange relation,

$$R_{12}^{\frac{1}{2}\frac{1}{2}}(\lambda) L_{13}^S(\lambda + \mu) L_{23}^S(\mu) = L_{23}^S(\mu) L_{13}^S(\lambda + \mu) R_{12}^{\frac{1}{2}\frac{1}{2}}(\lambda) \tag{5}$$

where $R^{\frac{1}{2}\frac{1}{2}}(\lambda) = L^{\frac{1}{2}}\left(\lambda + \frac{i}{2}\right)$ (up to a normalization, see below). The lower indices in (5) specify components of the tensor product $V_{\frac{1}{2}} \otimes V_{\frac{1}{2}} \otimes V_S$ where the R-matrix and the L-operators act. Actually, the R-matrix in (5) is a particular representation of a more general object, $R(\lambda) \in [U_q(sl_2)]^{\otimes 2}$, that satisfies the Yang–Baxter equation

$$R_{12}(\lambda)R_{13}(\lambda + \mu)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda + \mu)R_{12}(\lambda). \tag{6}$$

For a homogeneous spin chain or a related lattice model, it suffices to consider the 'diagonal' solution $R^{SS}(\lambda)$ of (6). However, to deal with an inhomogeneous chain, it is unavoidable to use the general solution $R^{S_1 S_2}(\lambda)$. We provide the necessary details below.

Define the 'total spin' operator $\mathbb{J}$ on $V_{S_1} \otimes V_{S_2}$ as

$$\mathbb{J} = \sum_{j=\delta S}^{S_1 + S_2} j \mathcal{P}_j \tag{7}$$

where $\delta S = |S_1 - S_2|$, and $\mathcal{P}_j$ is the projector onto V_j in the Clebsch–Gordan decomposition of $V_{S_1} \otimes V_{S_2}$. If $f(x)$ is a function non-singular on spec $\mathbb{J}$, then equation (7) allows us to define $f(\mathbb{J}) = \sum_{j=\delta S}^{S_1 + S_2} f(j)\mathcal{P}_j$. With this notation, a solution $R^{S_1 S_2}(\lambda)$ to the $U_q(sl_2)$-Yang–Baxter equation (6) consistent with the L-operator (4) can be written as follows (see appendix A):

$$R^{S_1 S_2}(\lambda) = (-1)^{\mathbb{J} - S_1 - S_2} q^{\epsilon(S_1(S_1+1) + S_2(S_2+1) - \mathbb{J}(\mathbb{J}+1))}$$

$$\times \frac{\Gamma_q(\delta S + 1 + i\lambda)}{\Gamma_q(\delta S + 1 - i\lambda)} \frac{\Gamma_q(\mathbb{J} + 1 - i\lambda)}{\Gamma_q(\mathbb{J} + 1 + i\lambda)} \left(R_\epsilon^{S_1 S_2}\right)^{-1} \qquad \epsilon = \pm. \tag{8}$$

Here $\Gamma_q(x)$ is a q-deformation of the gamma function such that[4]

$$\Gamma_q(x + 1) = [x]_q \Gamma_q(x) \tag{9}$$

and $R_\pm$ are the constant universal R-matrices [5, 47],

$$R_\epsilon = q^{-\epsilon S^3 \otimes S^3} \sum_{n=0}^{\infty} \frac{q^{\epsilon \frac{1}{2}(n - n^2)}}{\prod_{k=1}^{n}[k]_q} (\epsilon(q^{-1} - q)S^\epsilon \otimes S^{-\epsilon})^n q^{-\epsilon S^3 \otimes S^3}. \tag{10}$$

The $q = 1$ limit of (8) gives the well-known sl_2 solution constructed in [27].

As we discuss in appendix A, the following relation holds,

$$(-1)^{\mathbb{J} - 2S} q^{\epsilon(2S(S+1) - \mathbb{J}(\mathbb{J}+1))} \left(R_\epsilon^{SS}\right)^{-1} = \mathbb{P} \tag{11}$$

where $\mathbb{P}$ is the permutation in $V_S \otimes V_S$. Identity (11) ensures an important property of the solution (8):

$$R^{SS}(0) = \mathbb{P}. \tag{12}$$

Also, one can show that (8) enjoys the following symmetries,

$$R_{1/q}(\lambda) = (R_q(-\lambda))^{-1} = (R_q(\lambda))^t \tag{13}$$

$$R_{1/q}(0) = (R_q(0))^{-1} = (R_q(0))^t = R_q(0) \tag{14}$$

where t stands for the matrix transposition consistent with (3).

In order to obtain $R^{S_1 S_2}(\lambda)$ in terms of the algebra generators, which is needed, e.g., for constructing related Hamiltonians, one can substitute into (8) the following explicit form of the projectors (see, e.g., [28]),

$$\mathcal{P}_j = \prod_{\substack{l=\delta S \\ l \neq j}}^{S_1 + S_2} \frac{\mathbb{X}^{S_1 S_2} - [l]_q[l+1]_q}{[j-l]_q[j+l+1]_q} \tag{15}$$

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[4] It should be stressed here that equation (9) alone does not define the q-gamma function uniquely. However, we need to deal only with expressions of the form $\Gamma_q(x + n)/\Gamma_q(x)$, $n \in \mathbb{N}$, that are well defined.

where

$$\mathbb{X}^{S_1 S_2} = \left(q^{S^3} \otimes 1\right)(S^+ \otimes S^- + S^- \otimes S^+)\left(1 \otimes q^{-S^3}\right) + \frac{\cos \gamma}{2 \sin^2 \gamma}\left(1 \otimes 1 + q^{2S^3} \otimes q^{-2S^3}\right)$$

$$- \frac{1}{2 \sin^2 \gamma}\left(\left(1 \otimes q^{-2S^3}\right) \cos \gamma (2S_1 + 1) + \left(q^{2S^3} \otimes 1\right) \cos \gamma (2S_2 + 1)\right). \qquad (16)$$

Formulae (8), (15) and (16) allow us, in particular, to verify that $R^{\frac{1}{2}S}(\lambda)$ does reproduce the L-operator (4). Let us, however, describe another way of doing that. To this end we will construct the 'Baxterized' form of $R(\lambda)$ for higher spins. As we show in appendix A, the projectors admit the following form:

$$\mathcal{P}_j = \prod_{\substack{l = \delta S \\ l \neq j}}^{S_1 + S_2} \frac{q^{2l(l+1)} - q^{2S_1(S_1+1) + 2S_2(S_2+1)} R_+^{-1} R_-}{q^{2l(l+1)} - q^{2j(j+1)}}. \qquad (17)$$

This formula along with (8) allows us to find an expression for $R(\lambda)$ in terms of R_+ and R_-:

$$R^{\frac{1}{2}S}(\lambda) = \frac{(q - q^{-1})^{-1}}{[S + \frac{1}{2} + i\lambda]_q}\left[q^{\frac{1}{2} - i\lambda}\left(R_+^{\frac{1}{2}S}\right)^{-1} - q^{i\lambda - \frac{1}{2}}\left(R_-^{\frac{1}{2}S}\right)^{-1}\right] \qquad (18)$$

$$R^{1S}(\lambda) = \frac{(q - q^{-1})^{-2}}{[S + i\lambda]_q[S + 1 + i\lambda]_q}\left[(q^{2i\lambda - 1} - q^{-2}\varpi)(R_-^{1S})^{-1} + q^4\varpi\left(R_+^{1S}\right)^{-1} R_-^{1S}\left(R_+^{1S}\right)^{-1}\right.$$

$$\left. + (q^{1 - 2i\lambda} + (1 - q^2)\varpi - q^2\varpi^{-1})\left(R_+^{1S}\right)^{-1}\right] \qquad \varpi \equiv \frac{[2S + 1]_q}{[4S + 2]_q} \qquad (19)$$

etc. From (18) one easily verifies that $[S + 1 + i\lambda]_q R^{\frac{1}{2}S}\left(\lambda - \frac{i}{2}\right)$ coincides with $L^S(\lambda)$ given by (4).

Let us remark that, twisting (4), we can obtain other L-operators related to the XXZ chain. In particular, the following one is often used in the literature:

$$\hat{L}(\lambda) = e^{\gamma(\lambda - \frac{i}{2})S^3} L(\lambda) e^{\gamma\left(\frac{i}{2} - \lambda\right)S^3} = \begin{pmatrix} [S^3 - i\lambda]_q & S^- \\ S^+ & -[S^3 + i\lambda]_q \end{pmatrix}. \qquad (20)$$

From the Yang–Baxter equation (6) and the $U(1)$ symmetry of the solution (8)

$$\left[R_{ab}(\lambda), S_a^3 + S_b^3\right] = 0 \qquad (21)$$

it follows that the universal R-matrix corresponding to (20) is related to (8) as:

$$\hat{R}(\lambda) = e^{\gamma\lambda(1 \otimes S^3)} R(\lambda) e^{-\gamma\lambda(1 \otimes S^3)}. \qquad (22)$$

It has the following symmetries (see appendix A):

$$\hat{R}(-\lambda) = (\hat{R}(\lambda))^{-1} \qquad \hat{R}_{1/q}(-\lambda) = \hat{R}_q(\lambda) \qquad (\hat{R}(\lambda))^t = \hat{R}(\lambda). \qquad (23)$$

2.2. Hamiltonian

In order to construct a Hamiltonian for a closed alternating spin chain we apply a generalization of the method developed in [29] and consider two transfer matrices[5]

$$\tau^{(i)}(\lambda) = \text{tr}_a T_a^{(i)}(\lambda) = \text{tr}_a\left(R_{a,N}^{S_i S_2}(\lambda) R_{a,N-1}^{S_i S_1}(\lambda) \cdots R_{a,2}^{S_i S_2}(\lambda) R_{a,1}^{S_i S_1}(\lambda)\right) \qquad i = 1, 2 \qquad (24)$$

where the auxiliary space (denoted by the subscript a) is V_{S_1} and V_{S_2}, respectively. Since the monodromy matrices satisfy the Yang–Baxter equation

[5] If we introduce proper inhomogeneities of the spectral parameter, we will obtain massless relativistic dispersion relations for the particle-like excitations of the model [25].

$$R_{ab}^{S_i S_j}(\lambda - \mu) T_a^{(i)}(\lambda) T_b^{(j)}(\mu) = T_b^{(j)}(\mu) T_a^{(i)}(\lambda) R_{ab}^{S_i S_j}(\lambda - \mu) \tag{25}$$

the corresponding transfer matrices commute

$$[\tau^{(i)}(\lambda), \tau^{(j)}(\mu)] = 0. \tag{26}$$

Therefore, the first and higher order logarithmic derivatives of $\tau^{(i)}(\lambda)$ at $\lambda = 0$ are quantum integrals of motion. Moreover, thanks to the property (12), these integrals are local, according to theorem 3 in [30].

Using the property (12) (along with $\mathrm{tr}_a \mathbb{P}_{ab} = 1$), we find that

$$\tau^{(1)}(0) = \left(\prod_{\substack{k\text{-odd}}}^{N-3} \mathbb{P}_{k,k+2} \right) \left(\prod_{\substack{k\text{-odd}}}^{N-1} R_{k,k+1}^{S_1 S_2}(0) \right) \qquad \tau^{(2)}(0) = \left(\prod_{\substack{k\text{-even}}}^{N-2} \mathbb{P}_{k,k+2} \right) \left(\prod_{\substack{k\text{-even}}}^{N} R_{k,k+1}^{S_2 S_1}(0) \right) \tag{27}$$

where $N + 1 \equiv 1$ (the periodic boundary conditions). Noticing that for both $\tau(0)$ in (27) all $R(0)$ commute among each over, it is not difficult to obtain the following expressions for Hamiltonians,

$$\mathcal{H}^{(1)} = \frac{i}{2} \partial_\lambda \ln \tau^{(1)}(\lambda)|_{\lambda=0} = \sum_{n\text{-odd}}^{N-1} H_n^{(1)} \qquad \mathcal{H}^{(2)} = \frac{i}{2} \partial_\lambda \ln \tau^{(2)}(\lambda)|_{\lambda=0} = \sum_{n\text{-even}}^{N} H_n^{(2)} \tag{28}$$

with $H_n^{(i)}$ being local Hamiltonians involving three nearest sites:

$$H_n^{(i)} = \left(R_{n,n+1}^{S_i S_{\bar{\imath}}}(0) \right)^{-1} \left[\frac{i}{2} \partial_\lambda R_{n,n+1}^{S_i S_{\bar{\imath}}}(\lambda) \Big|_{\lambda=0} + h_{n,n+2}^{S_i S_i} R_{n,n+1}^{S_i S_{\bar{\imath}}}(0) \right] \tag{29}$$

where $\tilde{\imath} = i + 1 \pmod 2$, and

$$h_{nm}^{SS} = \frac{i}{2} \mathbb{P}_{nm} \partial_\lambda R_{nm}^{SS}(\lambda) \Big|_{\lambda=0} \tag{30}$$

is the local Hamiltonian of the homogeneous spin S chain. These formulae provide a particular case of local Hamiltonians for inhomogeneous chains derived in [30] (see also [19]). Obviously, they recover the homogeneous case as well since for $S_1 = S_2$ we have $R_{n,n+1}^{S_1 S_2}(0) = \mathbb{P}_{n,n+1}$, and then $H_n^{(i)}$ becomes a sum of two terms of the type (30).

Let us note that computing in the same way the local Hamiltonians $\hat{H}_n^{(i)}$ corresponding to the R-matrix (22) and taking the property (21) into account, we obtain

$$\hat{H}_n^{(i)} = H_n^{(i)} + \frac{i\gamma}{2} \left(S_n^3 - S_{n+2}^3 \right). \tag{31}$$

Hence for the total Hamiltonians we have $\hat{\mathcal{H}}^{(i)} = \mathcal{H}^{(i)}$.

Let $\Psi_q(x)$ denote the logarithmic derivative of the q-gamma function used in (8), so that we have, in particular, $\Psi_q(x + 1) = \Psi_q(x) + \gamma \cot \gamma x$. Then (8) allows us to rewrite the local Hamiltonian in the following, more explicit, form,

$$H_n^{(i)} = (R_\epsilon)_{n,n+1} q^{\epsilon \mathbb{J}_{n,n+1}(\mathbb{J}_{n,n+1}+1)} (-1)^{\mathbb{J}_{n,n+1}} [\Psi_q(\mathbb{J}_{n,n+1} + 1) - \Psi_q(\delta S + 1)$$
$$+ \Psi_q(\mathbb{J}_{n,n+2} + 1) - \Psi_q(1)] (-1)^{\mathbb{J}_{n,n+1}} q^{-\epsilon \mathbb{J}_{n,n+1}(\mathbb{J}_{n,n+1}+1)} \left(R_\epsilon^{-1} \right)_{n,n+1} \tag{32}$$

where the quantum space is V_{S_i} at the sites n, $n + 2$, and $V_{S_{\bar{\imath}}}$ at the site $n + 1$.

In order to exemplify (29) and (32), let us construct $H_n^{(1)}$ explicitly for $(S_1, S_2) = \left(\frac{1}{2}, S \right)$. Consider the XXX case first. Here we need only the following projectors,

$$\left(\mathcal{P}_{S \pm \frac{1}{2}}^{\frac{1}{2} S} \right)_{n,m} = \frac{1}{2S + 1} \left(S + \frac{1}{2} \pm \frac{1}{2} \pm \vec{\sigma}_n \cdot \vec{S}_m \right) \tag{33}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ consists of the Pauli matrices. In this case (32) yields

$$H_n^{(1)} = \frac{1}{(2S+1)^2}\left(2(\vec{\sigma}_n \cdot \vec{S}_{n+1}) + 2(\vec{S}_{n+1} \cdot \vec{\sigma}_{n+2})\right.$$
$$\left. + \{(\vec{\sigma}_n \cdot \vec{S}_{n+1}), (\vec{S}_{n+1} \cdot \vec{\sigma}_{n+2})\} + \left(\frac{1}{4} - S(S+1)\right)(\vec{\sigma}_n \cdot \vec{\sigma}_{n+2}) + \frac{3}{4}\right). \tag{34}$$

Here and below $\{\,,\,\}$ stands for the anticommutator. Since $\{(\vec{\sigma} \cdot \vec{a}), (\vec{\sigma} \cdot \vec{b})\} = 2(\vec{a} \cdot \vec{b})$, it is easy to see that, for $S = \frac{1}{2}$, equation (34) reduces to the usual Heisenberg Hamiltonian. In a slightly different form (34) was also given in [20].

In the XXZ case, it is easier to compute $\hat{H}_n^{(1)}$, taking into account the symmetry (23). To write down the answer in a compact form, we introduce the Sklyanin generators [31] of $U_q(sl_2)$,

$$\mathfrak{S}^0 = \left[\frac{1}{2}\right]_q \cos(\gamma S_3) \qquad \mathfrak{S}^1 = \frac{1}{2}(S^+ + S^-)$$
$$\mathfrak{S}^2 = \frac{1}{2i}(S^+ - S^-) \qquad \mathfrak{S}^3 = \cos\left(\frac{\gamma}{2}\right)[S_3]_q. \tag{35}$$

They satisfy the following quadratic relations:

$$[\mathfrak{S}^3, S^\pm] = \pm\{\mathfrak{S}^0, S^\pm\} \qquad [\mathfrak{S}^0, S^\pm] = \mp\left(\tan\frac{\gamma}{2}\right)^2\{\mathfrak{S}^3, S^\pm\}. \tag{36}$$

Using these relations, we obtain from (29)–(31) the following $\left(\frac{1}{2}, S\right)$-Hamiltonian (see also [19] for a $\left(\frac{1}{2}, 1\right)$-Hamiltonian),

$$\hat{H}_n^{(1)} = \frac{1}{4\left[S + \frac{1}{2}\right]_q^2}\left(2\{\mathfrak{S}_{n+1}^0, (\vec{\sigma}_n^\gamma \cdot \vec{\mathfrak{S}}_{n+1}) + (\vec{\mathfrak{S}}_{n+1} \cdot \vec{\sigma}_{n+2}^\gamma)\} + \cos\left(2\gamma S_n^3\right)\right.$$
$$+ \{(\vec{\sigma}_n \cdot \vec{\mathfrak{S}}_{n+1}), (\vec{\mathfrak{S}}_{n+1} \cdot \vec{\sigma}_{n+2}^\gamma)\} + \left((\mathfrak{S}_{n+1}^0)^2 - \vec{\mathfrak{S}}_{n+1} \cdot \vec{\mathfrak{S}}_{n+1}\right)(\vec{\sigma}_n \cdot \vec{\sigma}_{n+2}^\gamma)$$
$$\left. + \frac{3}{4}\cos\gamma + \left[\mathfrak{S}_{n+1}^0, \left[(\vec{\sigma}_n \cdot \vec{S}_{n+1}), (\vec{S}_{n+1} \cdot \vec{\sigma}_{n+2}^\gamma)\right]\right]\right) \tag{37}$$

where $\vec{\mathfrak{S}} = (\mathfrak{S}^1, \mathfrak{S}^2, \mathfrak{S}^3)$ and $\vec{\sigma}^\gamma = (\sigma^1, \sigma^2, \sigma^3 \cos\gamma)$. For $S = \frac{1}{2}$ we have $\mathfrak{S}^0 = \frac{1}{2}, \vec{\mathfrak{S}} = \frac{1}{2}\vec{\sigma}$ and then (37) reduces to the usual XXZ-deformation of the Heisenberg Hamiltonian.

Consider the $*$-structure on $U_q(sl_2)$ corresponding to the compact real form $U_q(su(2))$, i.e., an anti-automorphism such that

$$(S^\pm)^* = S^\mp \qquad (S^3)^* = S^3 \tag{38}$$

and that extends onto a tensor product as $(\xi \otimes \zeta)^* = \xi^* \otimes \zeta^*$. If q is real or $|q| = 1$ we have (see appendix A) the following 'unitarity' relations:

$$(R_q(\lambda))^* = (R_{\bar{q}}(\bar{\lambda}))^{-1} \qquad (\hat{R}_q(\lambda))^* = (\hat{R}_q(\bar{\lambda}))^{-1}. \tag{39}$$

Applying the latter relation along with (14) to the counterpart of (29) for $\hat{H}_n^{(i)}$ and taking into account that $h_{nm}^* = h_{nm}$ (see, e.g., [28]), we infer that

$$\left(\hat{H}_n^{(i)}\right)^* = \hat{H}_n^{(i)}. \tag{40}$$

Therefore, the total Hamiltonian of an alternating spin chain $\mathcal{H}^{(i)}$ is Hermitian if $\gamma \in \mathbb{R}$ or $i\gamma \in \mathbb{R}$.

2.3. Momentum operator

Unlike the homogeneous case, the transfer matrix $\tau^{(i)}(0)$ is not the shift operator and hence its logarithm cannot be interpreted as the momentum operator. Observe, however, that the products of permutations entering (27),

$$U^{(1)} = \prod_{k\text{-odd}}^{N-3} \mathbb{P}_{k,k+2} \qquad U^{(2)} = \prod_{k\text{-even}}^{N-2} \mathbb{P}_{k,k+2} \tag{41}$$

are shift operators of odd/even lattice sites by two steps, i.e.,

$$U^{(i)} \xi_n = \xi_{n+2} U^{(i)} \qquad \text{if} \quad n \equiv i \bmod 2. \tag{42}$$

This motivates us to consider quantum integrals of the motion generated by $\tau^{1,2}(\lambda) = \tau^{(1)}(\lambda)\tau^{(2)}(\lambda)$. Due to (26), the corresponding Hamiltonian is given by

$$\mathcal{H} = i\frac{J}{4}\partial_\lambda \ln \tau^{(1,2)}(\lambda)|_{\lambda=0} = \frac{J}{2}(\mathcal{H}^{(1)} + \mathcal{H}^{(2)}) \tag{43}$$

with $\mathcal{H}^{(i)}$ defined in (28) and $J > 0$ being a coupling constant.

In order to compute $\ln \tau^{1,2}(0)$ we need a property of the R-matrix that follows from evaluating the Yang–Baxter equation (6) in $V_{S_1} \otimes V_{S_2} \otimes V_{S_1}$ at $\mu = -\lambda$. In view of (12) this leads to the relation

$$R_{12}^{S_1 S_2}(\lambda) R_{21}^{S_2 S_1}(-\lambda) = R_{23}^{S_2 S_1}(-\lambda) R_{32}^{S_1 S_2}(\lambda) \tag{44}$$

and we infer, in particular, that

$$R_{12}^{S_1 S_2}(0) R_{21}^{S_2 S_1}(0) = 1 \otimes 1 \tag{45}$$

if the normalization of the R-matrix was chosen appropriately (as, e.g., in (8)).

Now, using (27), (42) and (45), we obtain

$$\tau^{1,2}(0) = U^{(1)} U^{(2)} \tag{46}$$

which generates a shift by two lattice sites. Thus, $\mathcal{P} = \frac{1}{2i} \ln \tau^{1,2}(0)$ can be regarded as the momentum operator (as was observed also in [19]). Similar to the homogeneous spin chain, the pair $(\mathcal{H}, \mathcal{P})$ defines in the gapless regime ($|q| = 1$) at zero temperature a conformally invariant system.

2.4. Bethe equations

Consider the alternating XXZ chain of even length N. According to the general scheme of the algebraic Bethe ansatz [29, 30] (see also [32] for a review), eigenvectors and eigenvalues of the auxiliary transfer matrix

$$\tau^{(0)}(\lambda) = \mathrm{tr}_a T_a^{(0)}(\lambda) = \mathrm{tr}_a\big(L_N^{S_2}(\lambda) L_{N-1}^{S_1}(\lambda) \cdots L_2^{S_2}(\lambda) L_1^{S_1}(\lambda)\big) \tag{47}$$

where the trace is taken over the auxiliary space $V_{\frac{1}{2}}$, are parametrized by solutions of the system of equations

$$\prod_{j=1}^{2} \left(\frac{\sinh \gamma (\lambda_\alpha + iS_j)}{\sinh \gamma (\lambda_\alpha - iS_j)} \right)^{\frac{N}{2}} = -\prod_{\beta=1}^{M} \frac{\sinh \gamma (\lambda_\alpha - \lambda_\beta + i)}{\sinh \gamma (\lambda_\alpha - \lambda_\beta - i)} \qquad \alpha = 1, \ldots, M \tag{48}$$

where M is the level of a Bethe vector $\Psi(\lambda_1, \ldots, \lambda_M)$; it is related to the total z-component of the spin as

$$\left(\sum_{n=1}^{N} S_n^3 \right) \Psi(\lambda_1, \ldots, \lambda_M) = S_z \Psi(\lambda_1, \ldots, \lambda_M) \tag{49}$$

where $S_z = ((S_1 + S_2)N/2 - M)$. The Yang–Baxter equation

$$R_{ab}^{\frac{1}{2}S_j}(\lambda - \mu)T_a^{(0)}(\lambda)T_b^{(j)}(\mu) = T_b^{(j)}(\mu)T_a^{(0)}(\lambda)R_{ab}^{\frac{1}{2}S_j}(\lambda - \mu) \qquad j = 1, 2 \qquad (50)$$

implies commutativity of the corresponding transfer matrices,

$$[\tau^{(0)}(\lambda), \tau^{(j)}(\mu)] = 0. \qquad (51)$$

Therefore the eigenvectors of $\tau^{(0)}(\lambda)$ are simultaneously eigenvectors for $\tau^{(j)}(\lambda)$. Moreover, as follows from the results of [30], eigenvalues of the momentum operator $\mathcal{P}$ and the Hamiltonian $\mathcal{H}$ (in the presence of an external magnetic field H) are given by

$$P(\lambda_1, \ldots, \lambda_M) = \sum_{\alpha=1}^{M} \sum_{j=1}^{2} p(\lambda_\alpha | S_j) \qquad E(\lambda_1, \ldots, \lambda_M) = -\frac{J}{2} \sum_{\alpha=1}^{M} \sum_{j=1}^{2} p'(\lambda_\alpha | S_j) - H S_z \qquad (52)$$

where we introduced the functions

$$p(\lambda | S) = \frac{1}{2i} \ln \frac{\sinh \gamma(\lambda - iS)}{\sinh \gamma(\lambda + iS)} = \arctan(\tanh(\gamma\lambda) \cot(\gamma S)) \qquad (53)$$

$$p'(\lambda | S) = \partial_\lambda p(\lambda | S) = \frac{\gamma \sin(2\gamma S)}{\cosh(2\gamma\lambda) - \cos(2\gamma S)}. \qquad (54)$$

The corresponding values for the alternating XXX chain are obtained by the $\gamma \to 0$ limit of (52)–(54).

3. Ground-state energy and finite-size effects at $T = 0$

3.1. Ground-state energy

Let π/γ be an integer. In the thermodynamic limit, when $M, N \to \infty$, the string hypothesis holds [33], namely solutions of the Bethe equations (48) form string-like clusters, i.e., groups of the type

$$\lambda_\alpha^{(n,k)} = \lambda_\alpha^n + \frac{i}{2}(n + 1 - 2k) \qquad k = 1, 2, \ldots, n$$
$$\lambda_\alpha^{(0)} = \lambda_\alpha^0 + i\frac{\pi}{2\gamma} \qquad (55)$$

where the centres of strings λ_α^n and λ_α^0 are real, and $\lambda^{(0)}$ is the negative parity string. The number n is called the length of a string.

At the zero temperature the system is in its ground state which, as we will see below, is a Dirac sea filled by strings of length $2S_1$ and $2S_2$ (negative parity strings are not present). Taking the logarithm of (48) and carrying out summation along the strings in the standard way, we obtain a set of equations on the centres of these strings:

$$\frac{N}{2} \left(\sum_{m=\frac{1}{2}}^{2S_i - \frac{1}{2}} p(\lambda_\alpha^{(i)} | m) + \sum_{m=\delta S + \frac{1}{2}}^{S_1 + S_2 - \frac{1}{2}} p(\lambda_\alpha^{(i)} | m) \right) = \pi Q_\alpha^{(i)} + \sum_{j=1}^{2} \sum_{\beta=1}^{M_j} \Phi_{2S_i, 2S_j}(\lambda_\alpha^{(i)} - \lambda_\beta^{(j)}) \qquad (56)$$

where $\delta S = |S_1 - S_2|$ and $\lambda_\alpha^{(i)} \equiv \lambda_\alpha^{2S_i}$, $i = 1, 2$. The numbers $Q_\alpha^{(i)}$ are distinct integer or half-integer numbers such that $|Q_\alpha^{(i)}| \leqslant \frac{N}{8} - \frac{1}{2}$, and

$$\Phi_{n,m}(\lambda) = \sum_{l=\frac{1}{2}|n-m|}^{\frac{1}{2}(n+m)-1} (p(\lambda | l) + p(\lambda | l + 1)) \qquad (57)$$

(for $n = m$ the term with $l = 0$ is omitted).

When $N, M \to \infty$, (56) become integral equations:

$$\sum_{m=\frac{1}{2}}^{2S_i - \frac{1}{2}} p'(\lambda|m) + \sum_{m=\delta S + \frac{1}{2}}^{S_1 + S_2 - \frac{1}{2}} p'(\lambda|m) = 2\pi \rho^{(i)}(\lambda) + 2 \sum_{j=1}^{2} \left(\Phi'_{2S_i, 2S_j} * \rho^{(j)} \right)(\lambda) \quad (58)$$

where the convolution is defined as $(u * v)(\lambda) = \int_{-\infty}^{\infty} d\mu \, u(\lambda - \mu) v(\mu)$, and $\Phi'_{n,m}(\lambda)$ is the derivative of (57). It is straightforward to check that

$$\int_{-\infty}^{\infty} d\lambda \, e^{ik\lambda} p'(\lambda|m) = \pi \frac{\sinh\left(\frac{k\pi}{2\gamma} - mk \right)}{\sinh \frac{k\pi}{2\gamma}} \qquad \text{if} \quad m < \frac{\pi}{\gamma}. \quad (59)$$

Using this and taking the Fourier transform of (58), it is not difficult to see that the only solution of these equations is given by

$$\rho^{(1)} = \rho^{(2)} = \frac{1}{4 \cosh \pi \lambda} \qquad \text{if} \quad 2\gamma \max(S_1, S_2) < \pi. \quad (60)$$

As follows from (52), (54) and (60), computation of the ground-state energy amounts to computing a sum of integrals of the following type,

$$I_\gamma(m) = \int_{-\infty}^{\infty} d\lambda \frac{p'(\lambda|m)}{\cosh \pi \lambda} = \int_{0}^{\infty} dk \frac{1}{\cosh \frac{k}{2}} \frac{\sinh\left(\frac{k\pi}{2\gamma} - mk \right)}{\sinh \frac{k\pi}{2\gamma}} \quad (61)$$

where, in the last equality, we took (59) into account. Expanding the functions in the denominator in the series, we obtain

$$I_\gamma(m) = \Psi\left(\frac{m}{2} + \frac{3}{4} \right) - \Psi\left(\frac{m}{2} + \frac{1}{4} \right) + \sum_{n \geq 1} \left[\Psi\left(\frac{m}{2} + \frac{3}{4} + \frac{\pi n}{2\gamma} \right) - \Psi\left(\frac{m}{2} + \frac{1}{4} + \frac{\pi n}{2\gamma} \right) \right.$$
$$\left. - \Psi\left(-\frac{m}{2} + \frac{3}{4} + \frac{\pi n}{2\gamma} \right) + \Psi\left(-\frac{m}{2} + \frac{1}{4} + \frac{\pi n}{2\gamma} \right) \right]. \quad (62)$$

Let us now show that $I_\gamma(m)$ admits, for real γ, a compact form in terms of a certain deformation of the Ψ-function. To this end we define the following special function,

$$\Gamma_q(x) = (q - q^{-1})^{-x} \frac{\Gamma_2(bx|b, b^{-1})}{\Gamma_2(b + b^{-1} - bx|b, b^{-1})} \quad (63)$$

where $q = e^{i\pi b^2}$, $0 < b < 1$, and $\Gamma_2(x|\omega_1, \omega_2)$ is the double gamma function introduced by Barnes [34]. Properties of $\Gamma_2(x|\omega_1, \omega_2)$ imply the following relations,

$$\Gamma_q(x + 1) = [x]_q \Gamma_q(x) \qquad (q - q^{-1})^{1/b^2} \Gamma_q(x + 1/b^2) = 2 \sin(\pi x) \Gamma_q(x) \quad (64)$$

the first of which allows us to regard $\Gamma_q(x)$ as a q-deformation of the gamma function. $\Gamma_q(x)$ is closely related to the noncompact quantum dilogarithm that has been actively studied recently [35] in the context of integrable models. In particular, it can be shown that

$$\ln \Gamma_q(x) = \text{const} - x \ln(q - q^{-1}) + \frac{\pi i}{2}(b + b^{-1} - bx)bx - \int_{\mathbb{R}+i0} \frac{dt}{t} \frac{e^{btx}}{(1 - e^{bt})(1 - e^{t/b})}. \quad (65)$$

This expression allows us to establish connection between the integral in (61) and the q-deformation of the Ψ-function defined via $\Psi_q(x) \equiv \partial_x \ln \Gamma_q(x)$. Namely, we find that

$$I_\gamma(m) = \Psi_{q^2}\left(\frac{m}{2} + \frac{3}{4} \right) - \Psi_{q^2}\left(\frac{m}{2} + \frac{1}{4} \right) \qquad \text{where} \quad q = e^{i\gamma}. \quad (66)$$

Now, from (52), (60), (61) and (66), we obtain the ground-state energy per lattice site:

$$e(S_1, S_2) = \frac{E}{N} = -\frac{J}{8} \sum_{i=1}^{2} \left(\sum_{m=\frac{1}{2}}^{2S_i - \frac{1}{2}} I_\gamma(m) + \sum_{m=\delta S + \frac{1}{2}}^{S_1 + S_2 - \frac{1}{2}} I_\gamma(m) \right)$$

$$= \frac{J}{4} \left[\Psi_{q^2} \left(\frac{1}{2} \delta S + \frac{1}{2} \right) + \Psi_{q^2} \left(\frac{1}{2} \right) - \Psi_{q^2} \left(\frac{1}{2}(S_1 + S_2) + \frac{1}{2} \right) \right]$$

$$- \frac{J}{8} \left[\Psi_{q^2} \left(S_1 + \frac{1}{2} \right) + \Psi_{q^2} \left(S_2 + \frac{1}{2} \right) \right]. \tag{67}$$

For a homogeneous chain, (67) reduces to

$$e(S) = \frac{J}{2} \left[\Psi_{q^2} \left(\frac{1}{2} \right) - \Psi_{q^2} \left(S + \frac{1}{2} \right) \right]. \tag{68}$$

Replacing q-deformed Ψ-functions with ordinary Ψ-functions in (67), (68), one recovers the energy densities in the XXX case. In particular, (68) turns in this limit into the well-known formula [12].

It is worth noticing that the first relation in (64) implies that $\Psi_{q^2}(x + 1) - \Psi_{q^2}(x) = 2\gamma \cot 2\gamma x$. Therefore, if S_1 and S_2 or, respectively, S are integer, we can rewrite (67) and (68) in terms of trigonometric functions:

$$e(S_1, S_2) = -\frac{\gamma J}{4} \left(\sum_{k=0}^{S_1 - 1} \cot \left[\gamma(2k + 1) \right] + \sum_{k=0}^{S_2 - 1} \cot \left[\gamma(2k + 1) \right] + 2 \sum_{k=\frac{1}{2}\delta S}^{\frac{1}{2}(S_1 + S_2) - 1} \cot \left[\gamma(2k + 1) \right] \right)$$

$$\tag{69}$$

$$e(S) = -\gamma J \sum_{k=0}^{S-1} \cot \left[\gamma(2k + 1) \right]. \tag{70}$$

3.2. Finite-size corrections

It is well known that statistical systems at criticality are expected to exhibit conformal invariance [36]. Therefore, the critical behaviour of such systems should be described by a certain conformal field theory. One can identify the corresponding effective central charge c by investigating the finite-size effects of the ground state of a critical system [37]. For a spin chain of a finite length N, the ground-state energy at zero temperature depends on N as follows:

$$e_N = e_\infty - \frac{\pi c v_s}{6N^2} + o(N^{-2}) \tag{71}$$

where v_s is the speed of sound. With the energy and momentum normalization as in (52) we have $v_s = \pi J/2$, provided that the condition in (60) is satisfied [16, 19, 24]. In this section we choose $J = 1$, so that $v_s = \pi/2$.

We will apply formula (71) to the alternating XXX spin chain in order to find the corresponding central charge numerically. To this end we start by solving numerically equation (56) in the case $\gamma = 0$. The solution of these equations is unique since they are extremum conditions for a convex function (Yang's functional, see [2] for the analogous consideration in the homogeneous case).

It is interesting to note that, if we assume that the roots of the Bethe equations form exact strings even for finite N and that the centres of these strings are given by (56), then formula (71) yields the same value $\tilde{c} = 2 + O(N^{-1})$ for all alternating chains (see table 1) thus

Table 1. Alternating XXX chains with $N = 16$ sites. $\lambda_\alpha^{S_i}$ are the real roots of equation (56) for $\gamma \to 0$; the estimate $\tilde{c}$ of the central charge is computed by (71) under the assumption that $\lambda_\alpha^{S_i}$ are centres of exact strings. λ_α^n are the complex roots of equation (48) for $\gamma \to 0$ corresponding to the genuine ground state; c is the corresponding finite-size estimate computed by (71).

S_1, S_2	$\lambda_\alpha^{S_1}$	$\lambda_\alpha^{S_2}$	$\tilde{c}$	λ_α^n	c
$\frac{1}{2}, 1$	$\pm0.129\,530$ $\pm0.538\,739$	$\pm0.131\,152$ $\pm0.557\,922$	2.08	$\pm0.543\,466, \pm0.129\,918,$ $\pm0.547\,454 \pm i0.504\,109,$ $\pm0.130\,894 \pm i0.500\,000$	2.06
$1, \frac{3}{2}$	$\pm0.130\,724$ $\pm0.554\,788$	$\pm0.131\,252$ $\pm0.561\,180$	2.10	$\pm0.549\,334 \pm i0.552\,225,$ $\pm0.130\,456 \pm i0.516\,086,$ $\pm0.562\,630, \pm0.561\,099 \pm i1.007\,705,$ $\pm0.131\,684, \pm0.131\,684 \pm i1.000\,000$	2.59
$\frac{1}{2}, \frac{3}{2}$	$\pm0.130\,351$ $\pm0.535\,032$	$\pm0.131\,852$ $\pm0.552\,153$	2.08	$\pm0.449\,421, \pm0.094\,976,$ $\pm0.668\,491, \pm0.544\,016 \pm i1.055\,667,$ $\pm0.168\,193, \pm0.131\,555 \pm i1.017\,436$	2.60

predicting the conformal anomaly $\tilde{c} = 2$ independent of the spins (provided that $S_1 \neq S_2$). The interpretation of this phenomenon is that the field theories arising as the $H = 0, T \to 0$ limit and the $T = 0, H \to 0$ limit of a spin chain are, in general, different (see the discussion in [22]), and a finite-size computation based on the string hypothesis corresponds to the latter limit. The same phenomenon occurs for a homogeneous chain; in this case an analytic or numeric finite-size computation using the string hypothesis gives $\tilde{c} = 1$ independent of spin S, whereas those not using the string hypothesis give $c = 3S/(S + 1)$. Comparing this fact with our result described above, we may conjecture that a finite-size computation based on the string hypothesis performed for a chain containing equal numbers of l different spins will give $\tilde{c} = l$ (which is the smallest possible value of c for such a chain, see section 5).

Although, as we discussed above, the string solutions constructed on the basis of (56) are not exact, they, as seen from table 1, provide good initial data for iterative numerical computations. Using them as a first approximation and applying a Newton-type method, we solved numerically the XXX version of equations (48) for several alternating chains and computed the corresponding ground-state energy. Then we used formula (71) and equation (67) in order to find the finite-size estimate of the central charge for these chains. Our results, presented in table 2, are described by the following expression:

$$c = \frac{3S_1}{S_1 + 1} + \frac{3(S_2 - S_1)}{S_2 - S_1 + 1}. \tag{72}$$

This formula was conjectured in [20] as an extrapolation of the results for the $\left(\frac{1}{2}, S\right)$ XXX chain. Since it is known that for the homogeneous XXZ chain the central charge does not depend on the anisotropy (if $2\gamma S < \pi$) and coincides with that of the XXX chain, one can expect that (72) applies to the alternating XXZ chain as well (provided that the condition in (60) is satisfied). This is indeed so, as we will demonstrate below.

4. Thermodynamics

In this section we will investigate the thermodynamics of the alternating XXZ chain in the case where the anisotropy parameter $\nu = \frac{\pi}{\gamma} \geqslant 3$ is an integer number and $\nu > 2S_i$. Our analysis follows, with proper modifications, the standard Bethe ansatz method [9, 14, 16, 19, 32].

Table 2. Alternating XXX chains with N sites (the coupling constant is $J = 1$). e is the energy per site for the numerically found ground state solution of equation (48) for $\gamma \to 0$; c is the corresponding finite-size estimate of the central charge obtained by equation (71). The $N = \infty$ values are predicted by equations (67) and (72). The asterisks indicate lack of stable numerical results.

S_1, S_2	$N = 8$ $|e|$	c	$N = 16$ $|e|$	c	$N = 24$ $|e|$	c	$N = \infty$ $|e|$	c
$\frac{1}{2}, 1$	0.666 301	2.21	0.644 511	2.06	0.640 791	2.03	0.637 888	2
$1, \frac{3}{2}$	0.917 421	2.79	0.889 926	2.59	0.885 257	2.55	0.881 620	2.5
$\frac{3}{2}, 2$	1.086 631	3.14	1.055 557	2.91	1.050 305	2.86	1.046 222	2.8
$2, \frac{5}{2}$	1.213 910	3.39	1.180 345	3.12	1.174 695	3.07	1.170 311	3
$\frac{1}{2}, \frac{3}{2}$	0.661 489	2.84	0.633 361	2.60	0.628 649	2.56	0.625	2.5
$\frac{1}{2}, 2$	0.666 989	3.24	0.634 793	2.94	*		0.625 352	2.8
$\frac{1}{2}, \frac{5}{2}$	0.675 916	3.52	0.640 833	3.17	*		0.630 647	3

4.1. Bethe equations at $T > 0$

Denote $\vartheta_i \equiv 2S_i$; without loss of generality we will always assume that $\vartheta_2 \geqslant \vartheta_1$. Following [19, 25], it is convenient to introduce the functions

$$e_n(\lambda; \nu) \equiv \frac{\sinh \gamma \left(\lambda + \frac{in}{2}\right)}{\sinh \gamma \left(\lambda - \frac{in}{2}\right)} \qquad g_n(\lambda; \nu) \equiv e_n \left(\lambda \pm \frac{i\pi}{2\gamma}\right) = \frac{\cosh \gamma \left(\lambda + \frac{in}{2}\right)}{\cosh \gamma \left(\lambda - \frac{in}{2}\right)}. \tag{73}$$

Then the Bethe equations (48) acquire the form

$$e_{\vartheta_1}(\lambda_\alpha)^{N/2} e_{\vartheta_2}(\lambda_\alpha)^{N/2} = -\prod_{\beta=1}^{M} e_2(\lambda_\alpha - \lambda_\beta). \tag{74}$$

The spin, energy and momentum of a state are given in terms of the Bethe ansatz roots λ_α by equations (49) and (52). In this section we set $J = 1/\pi$.

In the thermodynamic limit $N, M \to \infty$ the string hypothesis holds [33], and the Bethe equations become

$$\prod_{j=1}^{2} X_{n\vartheta_j}\left(\lambda_\alpha^n\right)^{N/2} = (-)^n \prod_{m=1}^{\nu-1} \prod_{\beta=1}^{M_m} E_{nm}\left(\lambda_\alpha^n - \lambda_\beta^m\right) \prod_{\beta=1}^{M_0} G_{n1}\left(\lambda_\alpha^n - \lambda_\beta^0\right) \tag{75}$$

$$\prod_{j=1}^{2} g_{\vartheta_j}\left(\lambda_\alpha^0\right)^{N/2} = -\prod_{m=1}^{\nu-1} \prod_{\beta=1}^{M_m} G_{1m}\left(\lambda_\alpha^0 - \lambda_\beta^m\right) \prod_{\beta=1}^{M_0} e_2\left(\lambda_\alpha^0 - \lambda_\beta^0\right) \tag{76}$$

where the notation for the centres of strings is as in (55), n takes values $1, 2, \ldots, (\nu - 1)$, and

$$X_{nm}(\lambda) = e_{|n-m+1|}(\lambda) e_{|n-m+3|}(\lambda) \cdots e_{(n+m-3)}(\lambda) e_{(n+m-1)}(\lambda)$$
$$E_{nm}(\lambda) = e_{|n-m|}(\lambda) e_{|n-m+2|}^2(\lambda) \cdots e_{(n+m-2)}^2(\lambda) e_{(n+m)}(\lambda) \tag{77}$$
$$G_{nm}(\lambda) = g_{|n-m|}(\lambda) g_{|n-m+2|}^2(\lambda) \cdots g_{(n+m-2)}^2(\lambda) g_{(n+m)}(\lambda).$$

The energy (52) of a given state takes the form

$$E = -\frac{1}{2} \sum_{n=1}^{\nu-1} \sum_{\alpha=1}^{M_n} \left(Z_{n\vartheta_1}\left(\lambda_\alpha^n\right) + Z_{n\vartheta_2}\left(\lambda_\alpha^n\right)\right) - \frac{1}{2} \sum_{\alpha=1}^{M_0} \left(b_{\vartheta_1}\left(\lambda_\alpha^0\right) + b_{\vartheta_2}\left(\lambda_\alpha^0\right)\right) - H S_z \tag{78}$$

where

$$Z_{nm}(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln X_{nm}(\lambda) \qquad b_n(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln g_n(\lambda). \tag{79}$$

It is useful to note that the Fourier transforms of these quantities are

$$\hat{Z}_{nm}(\omega) = \frac{\sinh\left((\nu - \max(n, m))\frac{\omega}{2}\right) \sinh\left((\min(n, m))\frac{\omega}{2}\right)}{\sinh\left(\frac{\nu\omega}{2}\right) \sinh\left(\frac{\omega}{2}\right)} \tag{80}$$

$$\hat{b}_n(\omega; \nu) = -\frac{\sinh\left(\frac{n\omega}{2}\right)}{\sinh\left(\frac{\nu\omega}{2}\right)} \qquad 0 < n < \nu$$

$$= -\frac{\sinh\left((n - 2\nu)\frac{\omega}{2}\right)}{\sinh\left(\frac{\nu\omega}{2}\right)} \qquad \nu < n < 3\nu. \tag{81}$$

In the $N \to \infty$ limit we obtain from (78) an expression for the energy density:

$$e \equiv \frac{E}{N} = -\frac{1}{2} \sum_{n=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda \, (Z_{n\vartheta_1}(\lambda) + Z_{n\vartheta_2}(\lambda))\rho_n(\lambda) - \frac{1}{2} \int_{-\infty}^{\infty} d\lambda \, (b_{\vartheta_1}(\lambda) + b_{\vartheta_2}(\lambda))\rho_0(\lambda)$$

$$+ H \left(\sum_{n=1}^{\nu-1} n \int_{-\infty}^{\infty} d\lambda \, \rho_n(\lambda) + \int_{-\infty}^{\infty} d\lambda \, \rho_0(\lambda) - \frac{1}{4}(\vartheta_1 + \vartheta_2) \right) \tag{82}$$

where ρ_n is the density of the n-strings (pseudo-particles) and ρ_0 is the density of the negative parity string. The energy density of the ground state ($T = 0, H = 0$), which consists of two filled Dirac seas (strings of length ϑ_1 and ϑ_2), is given by

$$e_0 = \frac{E_0}{N} = -\frac{1}{4} \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} d\lambda \, Z_{\vartheta_i \vartheta_j}(\lambda) s(\lambda) \tag{83}$$

where (cf (60))

$$s(\lambda) = \frac{1}{2 \cosh(\pi\lambda)} \qquad \hat{s}(\omega) = \frac{1}{2 \cosh\frac{\omega}{2}}. \tag{84}$$

Besides the densities of pseudo-particles we need to introduce the densities of holes $\tilde{\rho}_n$ (for n-strings) and $\tilde{\rho}_0$ (for the negative parity string). Then, taking the logarithm of (75)–(76) and differentiating, one finds

$$\tilde{\rho}_n(\lambda) = \frac{1}{2}(Z_{n\vartheta_1}(\lambda) + Z_{n\vartheta_2}(\lambda)) - \sum_{m=1}^{\nu-1} A_{nm} * \rho_m(\lambda) - B_n * \rho_0(\lambda)$$

$$- (\rho_0(\lambda) + \tilde{\rho}_0(\lambda)) = \frac{1}{2}(b_{\vartheta_1} + b_{\vartheta_2}) - \sum_{m=1}^{\nu-1} B_m * \rho_m(\lambda) - a_2 * \rho_0(\lambda) \tag{85}$$

where

$$A_{nm}(\lambda) = \delta_{nm}\delta(\lambda) + \frac{i}{2\pi} \frac{d}{d\lambda} \ln E_{nm}(\lambda) \qquad B_n(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln G_{1n}(\lambda)$$

$$a_n(\lambda) = \frac{i}{2\pi} \frac{d}{d\lambda} \ln e_n(\lambda) \qquad \hat{A}_{nm}(\omega) = 2 \cosh\left(\frac{\omega}{2}\right) \hat{Z}_{nm}(\omega)$$

$$\hat{B}_n(\omega) = -\frac{2 \cosh(\frac{\omega}{2}) \sinh(\frac{n\omega}{2})}{\sinh(\frac{\nu\omega}{2})} + \delta_{n,\nu-1}. \tag{86}$$

In order to find ρ_n, ρ_0 in terms of $\tilde{\rho}_n$, $\tilde{\rho}_0$, it is convenient to consider the convolution of the first of the density equations (85) with the inverse of A_{nm}, which is given by

$$\hat{A}_{nm}^{-1} = \delta_{nm} - \hat{s}(\omega)(\delta_{n,m+1} + \delta_{n,m-1}). \tag{87}$$

Taking into account the following identities (summation over m is implied)

$$A_{nm}^{-1} * Z_{mk}(\lambda) = s(\lambda)\delta_{nk} \qquad A_{nm}^{-1} * m = \frac{\nu}{2}\delta_{n,\nu-1} \qquad A_{nm}^{-1} * B_m(\lambda) = -s(\lambda)\delta_{n,\nu-2} \tag{88}$$

$$a_2(\lambda) + s * B_{\nu-2}(\lambda) = 0 \qquad b_n(\lambda) + s(\lambda)\delta_{n,\nu-1} = -s * Z_{n,\nu-2}(\lambda) = s * B_n(\lambda) \tag{89}$$

we obtain the following expressions:

$$\rho_n(\lambda) = \frac{1}{2}s(\lambda)(\delta_{n\vartheta_1} + \delta_{n\vartheta_2}) - \sum_{m=1}^{\nu-1} A_{nm}^{-1} * \tilde{\rho}_m(\lambda) + \delta_{n,\nu-2}s * \rho_0(\lambda)$$

$$\rho_0(\lambda) = -\tilde{\rho}_0(\lambda) + s * \tilde{\rho}_{\nu-2}(\lambda) + \frac{1}{2}s(\lambda)(\delta_{\vartheta_1,\nu-1} + \delta_{\vartheta_2,\nu-1}). \tag{90}$$

According to the standard statistical principles, the entropy density of the system is given by

$$s \equiv \frac{S}{N} = \sum_{n=0}^{\nu-1} \int_{-\infty}^{\infty} d\lambda \left(\rho_n(\lambda) \ln\left(1 + \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}\right) + \tilde{\rho}_n(\lambda) \ln\left(1 + \frac{\rho_n(\lambda)}{\tilde{\rho}_n(\lambda)}\right) \right). \tag{91}$$

Denote $\eta_n(\lambda) = \frac{\tilde{\rho}_n(\lambda)}{\rho_n(\lambda)}$. The free energy of the system is $F = E - TS$. The equilibrium condition, $\delta F = 0$, along with relations (82), (85) and (91), yields the thermodynamic Bethe ansatz equations

$$T \ln\left(1 + \eta_n(\lambda)\right) = nH - \frac{1}{2}\left(Z_{n\vartheta_1}(\lambda) + Z_{n\vartheta_2}(\lambda)\right)$$

$$+ T \sum_{m=1}^{\nu-1} A_{nm} * \ln\left(1 + \eta_m^{-1}(\lambda)\right) - T B_n * \ln\left(1 + \eta_0^{-1}(\lambda)\right)$$

$$T \ln\left(\frac{1 + \eta_0(\lambda)}{1 + \eta_0^{-1}(\lambda)}\right) = H - \frac{1}{2}\left(b_{\vartheta_1}(\lambda) + b_{\vartheta_2}(\lambda)\right)$$

$$+ T \sum_{m=1}^{\nu-1} B_m * \ln\left(1 + \eta_m^{-1}(\lambda)\right) - T a_2 * \ln\left(1 + \eta_0^{-1}(\lambda)\right). \tag{92}$$

These equations can be rewritten with the help of (87)–(89) as follows

$$\epsilon_n(\lambda) = s(\lambda) * T \ln[(1 + \eta_{n+1}(\lambda))(1 + \eta_{n-1}(\lambda))] + \frac{\nu H}{2}\delta_{n,\nu-1}$$

$$+ \delta_{n,\nu-2}s(\lambda) * T \ln\left(1 + \eta_0^{-1}(\lambda)\right) - \frac{1}{2}s(\lambda)\left(\delta_{n\vartheta_1} + \delta_{n\vartheta_2}\right) \tag{93}$$

$$\epsilon_0(\lambda) = \frac{\nu H}{2} - s(\lambda) * T \ln(1 + \eta_{\nu-2}(\lambda)) + \frac{1}{2}s(\lambda)\left(\delta_{\vartheta_1,\nu-1} + \delta_{\vartheta_2,\nu-1}\right)$$

where $\epsilon_n(\lambda) \equiv T \ln\eta_n(\lambda)$ are the so-called pseudo-energies (the terms with $\ln(1 + \eta_0)$ and $\ln(1+\eta_\nu)$ are omitted in the equations for ϵ_1 and $\epsilon_{\nu-1}$, respectively). Observe that the equations for ϵ_0 and $\epsilon_{\nu-1}$ in (93) lead to the relation

$$\epsilon_0(\lambda) = \nu H - \epsilon_{\nu-1}(\lambda). \tag{94}$$

This implies, in particular, that the case $\vartheta_2 = \nu - 1$ (recall that $\vartheta_1 \leqslant \vartheta_2$) is special because then an extra term appears in the equation for ϵ_0. Indeed, if $\vartheta_2 = \nu - 1$, the ϑ_2-strings get

close to the odd parity strings, namely $\Im\left(\lambda_\beta^{(0)} - \lambda_\alpha^{(\vartheta_2,1)}\right) = 1$ (see (55)). As seen from the rhs of (48), this affects the structure of the roots of the Bethe equations.

Using (85), (92) and (89), we obtain the following expression for the free energy density at equilibrium:

$$
\begin{aligned}
f = \frac{F}{N} &= -\frac{T}{2} \sum_{n=1}^{\nu-1} \int_{-\infty}^{\infty} d\lambda \, \ln\left(1 + \eta_n^{-1}(\lambda)\right)\left(Z_{n\vartheta_1}(\lambda) + Z_{n\vartheta_2}(\lambda)\right) \\
&\quad + \frac{T}{2} \int_{-\infty}^{\infty} d\lambda \, \ln\left(1 + \eta_0^{-1}(\lambda)\right)\left(b_{\vartheta_1}(\lambda) + b_{\vartheta_2}(\lambda)\right) - \frac{(\vartheta_1 + \vartheta_2)H}{4} \\
&= e_0 - \frac{T}{2} \sum_{i=1}^{2} \left\{ \int_{-\infty}^{\infty} d\lambda \, s(\lambda) \ln\left(1 + \eta_{\vartheta_i}(\lambda)\right) + \delta_{\vartheta_i,\nu-1} \int_{-\infty}^{\infty} d\lambda \, s(\lambda) \ln\left(1 + \eta_0^{-1}(\lambda)\right) \right\}.
\end{aligned}
$$

(95)

4.2. The $T = 0$, $H \ll 1$ limit

At zero temperature, the ground state in the anti-ferromagnetic regime consists of two filled Dirac seas (strings of length ϑ_1 and $\vartheta_2 \geqslant \vartheta_1$). In this case we can rewrite equations (92) for the strings ϑ_1 and ϑ_2 only in the following approximate form:

$$
\epsilon_i(\lambda) = -\frac{1}{2} \sum_{j=1}^{2} Z_{\vartheta_i \vartheta_j}(\lambda) + \vartheta_i H + \sum_{j=1}^{2} \tilde{A}_{\vartheta_i \vartheta_j} * T \ln\left(1 + \eta_{\vartheta_j}^{-1}(\lambda)\right)
$$

$$
\tilde{A}_{nm}(\lambda) \equiv A_{nm}(\lambda) - \delta_{nm}\delta(\lambda).
$$

(96)

In the $T = 0$ limit the following quantities are defined for $i = 1, 2$:

$$
T \ln\left(1 + \eta_{\vartheta_i}^{\pm 1}\right) \to \pm\epsilon_i^{\pm} \qquad \epsilon_i^{\pm} = \tfrac{1}{2}(\epsilon_i \pm |\epsilon_i|).
$$

(97)

Then equation (96) can be rewritten as follows:

$$
\sum_{j=1}^{2} A_{\vartheta_i \vartheta_j} * \epsilon_j(\lambda) = -\frac{1}{2} \sum_{j=1}^{2} Z_{\vartheta_i \vartheta_j}(\lambda) + \vartheta_i H + \sum_{j=1}^{2} \tilde{A}_{\vartheta_i \vartheta_j} * \epsilon_j^{+}(\lambda).
$$

(98)

By solving the above system with the help of (88), we obtain

$$
\epsilon_i(\lambda) = -\frac{1}{2} s(\lambda) + \frac{1}{2} \delta_{i2} \xi H + M_{ij} * \epsilon_j^{+}(\lambda) \qquad \xi \equiv \frac{\nu}{\nu - \vartheta_2}
$$

(99)

where the kernel M has the following Fourier image,

$$
\hat{M}(\omega) = \begin{pmatrix} \hat{h}_1(\omega) & \hat{h}(\omega) \\ \hat{h}(\omega) & \hat{h}_2(\omega) \end{pmatrix}
$$

(100)

with

$$
\hat{h}_1(\omega) = \frac{\sinh(\delta\vartheta - 1)\frac{\omega}{2}}{2 \cosh \frac{\omega}{2} \sinh \delta\vartheta \frac{\omega}{2}} + \frac{\sinh(\vartheta_1 - 1)\frac{\omega}{2}}{2 \cosh \frac{\omega}{2} \sinh(\vartheta_1)\frac{\omega}{2}}
$$

$$
\hat{h}_2(\omega) = \frac{\sinh(\delta\vartheta - 1)\frac{\omega}{2}}{2 \cosh \frac{\omega}{2} \sinh \delta\vartheta \frac{\omega}{2}} + \frac{\sinh(\nu - \vartheta_2 - 1)\frac{\omega}{2}}{2 \cosh \frac{\omega}{2} \sinh(\nu - \vartheta_2)\frac{\omega}{2}} \qquad \hat{h}(\omega) = \frac{\sinh \frac{\omega}{2}}{2 \cosh \frac{\omega}{2} \sinh \delta\vartheta \frac{\omega}{2}}
$$

(101)

and $\delta\vartheta = \vartheta_2 - \vartheta_1$. Note that, as in [38], the quantities h_i, h are related to the LL (RR) scattering between left (right) movers, and the LR (left-right movers) massless scattering (see, e.g., [39]).

Equations (99) imply that $\epsilon_i(\lambda)$ are monotonically increasing functions of $|\lambda|$ with the maximal values given by

$$\epsilon_i(\infty) = \frac{1}{2}\xi H\left((1 - \hat{M}(0))^{-1}\binom{0}{1}\right)_i = \vartheta_i H. \tag{102}$$

Therefore, for H small but non-vanishing, the pseudo-energies $\epsilon_i(\lambda)$ are negative only on finite intervals $|\lambda| < \alpha_i$. Having (99) in mind, we introduce the zeros of the pseudo-energies as follows,

$$\alpha_i = -\frac{1}{\pi}(\ln \xi H + \ln k_i + O(1/\ln H)) \tag{103}$$

where k_i (which do not depend on H) are to be determined later. Next we define the functions

$$S_i(\lambda) = (\xi H)^{-1}\epsilon_i(\lambda + \alpha_i) \qquad \text{for} \quad \lambda \geqslant 0 \tag{104}$$

and $S_i(\lambda) = 0$ for $\lambda < 0$. Making the shift $\lambda \to \lambda + \alpha_i$ in (99) and using that $s(\lambda + \alpha_i) \to \xi H k_i e^{-\pi\lambda}$ (since $\alpha_i \to \infty$ as $H \to 0$), we obtain a Wiener–Hopf-type system

$$\begin{aligned}
S_1(\lambda) &= -\frac{k_1}{2}e^{-\pi\lambda} + \int_0^\infty d\lambda'\, h_1(\lambda - \lambda')S_1(\lambda') + \int_0^\infty d\lambda'\, h(\lambda - \lambda' + \alpha_1 - \alpha_2)S_2(\lambda') \\
S_2(\lambda) &= \frac{1}{2} - \frac{k_2}{2}e^{-\pi\lambda} + \int_0^\infty d\lambda'\, h(\lambda - \lambda' + \alpha_2 - \alpha_1)S_1(\lambda') + \int_0^\infty d\lambda'\, h_2(\lambda - \lambda')S_2(\lambda').
\end{aligned} \tag{105}$$

The above system can be solved via the standard Wiener–Hopf methods as follows. The Fourier image of the matrix kernel K of the system (105) is

$$\hat{K}(\omega) = \begin{pmatrix} \hat{h}_1(\omega) & e^{-i\omega(\alpha_1 - \alpha_2)}\hat{h}(\omega) \\ e^{-i\omega(\alpha_2 - \alpha_1)}\hat{h}(\omega) & \hat{h}_2(\omega) \end{pmatrix}. \tag{106}$$

Its resolvent admits the following factorization [40],

$$(1 - \hat{K}(\omega))^{-1} = G_+(\omega)G_-(\omega) \tag{107}$$

where $G_\pm(\omega)$ are analytic for $\pm\Im\omega \geqslant 0$, respectively, and also $G_+^t(\omega) = G_-(-\omega)$ for $\Im\omega \geqslant 0$. The solution of the system (105) (see also [21]) is

$$\hat{S}(\omega) = \frac{i}{2}\left(\frac{1}{\omega + i0} - \frac{1}{\omega + i\pi}\right)G_+(\omega)G_-(0)\binom{0}{1} \tag{108}$$

$$k = G_-^{-1}(-i\pi)G_-(0)\binom{0}{1}. \tag{109}$$

Denote $D = \det(1 - \hat{K}(0))$. From (106)–(109) we compute the following quantity:

$$k^t \hat{S}(i\pi) = \frac{1}{4\pi}(0\ 1)G_+(0)G_-(0)\binom{0}{1} = \frac{1}{4\pi}D^{-1}(1 - \hat{h}_1(0)) = \frac{\vartheta_2}{2\pi\xi}. \tag{110}$$

Recall that the odd parity string contributes to the free energy (95) only if $\vartheta_i = \nu - 1$. Observe, however, that its contribution vanishes as $T \to 0$ because it follows from (102) and (94) that $\epsilon_0(\lambda)$ is positive. Thus, as $T \to 0$, the free energy of the system becomes

$$\begin{aligned}
f &= e_0 - \sum_{i=1}^2 \int_{\alpha_i}^\infty d\lambda\, s(\lambda)\epsilon_i(\lambda) = e_0 - \xi H \sum_{i=1}^2 \int_0^\infty d\lambda\, s(\lambda + \alpha_i)S_i(\lambda) \\
&= e_0 - \xi^2 H^2 \sum_{i=1}^2 k_i \int_0^\infty d\lambda\, e^{-\pi\lambda}S_i(\lambda) = e_0 - \xi^2 H^2 k^t \hat{S}(i\pi).
\end{aligned} \tag{111}$$

Substituting here (110), we obtain the following expression for the magnetic field dependence of the free energy:

$$f = e_0 - \frac{1}{2\pi} \frac{\nu \vartheta_2}{\nu - \vartheta_2} H^2. \tag{112}$$

The magnetic susceptibility is given by $\chi = -\partial_H^2 f$. The value of $v_s \chi$ does not depend on the normalization of the Hamiltonian (v_s is the speed of sound, which in this section is $\frac{1}{2}$). Thus, we infer from (112) that

$$v_s \chi = \frac{1}{2\pi} \frac{\nu \vartheta_2}{\nu - \vartheta_2}. \tag{113}$$

Note that the contribution of the external magnetic field to the free energy (112) does not depend on $\delta\vartheta$, that is why our result coincides with the magnetization of the homogeneous XXZ chain found in [14] (where $v_s = J\pi/2$). Also, choosing $\vartheta_2 = 2$ in (113), we recover the results obtained for the alternating $\left(\frac{1}{2}, 1\right)$ chain both in the XXX [21] and XXZ [24] cases.

4.3. The $H = 0, T \ll 1$ limit

Here we will investigate the low temperature thermodynamics of the model in order to find the corresponding conformal anomaly. In order to evaluate the entropy of the system at small but nonzero temperature we use the following approximations which hold for $\lambda \gg 1$ (see, e.g., [14, 16]),

$$\rho_n(\lambda) = \frac{1}{\pi}(-1)^{\delta_{n0}} f_n(\lambda) \frac{d}{d\lambda} \epsilon_n(\lambda) \qquad \tilde{\rho}_n(\lambda) = \frac{1}{\pi}(-1)^{\delta_{n0}}(1 - f_n(\lambda)) \frac{d}{d\lambda} \epsilon_n(\lambda) \tag{114}$$

where

$$f_n(\lambda) = \left(1 + e^{\frac{\epsilon_n}{T}}\right)^{-1}. \tag{115}$$

Employing (114) and taking into account that $\epsilon_n(-\lambda) = \epsilon_n(\lambda)$, the entropy (91) can be written as

$$s = -\frac{2}{\pi} \sum_{n=0}^{\nu-1} (-1)^{\delta_{n0}} \int_{\epsilon_n(0)}^{\epsilon_n(\infty)} d\epsilon_n (f_n(\lambda) \ln f_n(\lambda) + (1 - f_n(\lambda) \ln(1 - f_n(\lambda)))$$

$$= \frac{4T}{\pi} \sum_{n=0}^{\nu-1} (-1)^{\delta_{n0}} \left(L\left(f_n^{(0)}\right) - L\left(f_n^{(\infty)}\right)\right) \tag{116}$$

where $f_n^{(0)} \equiv f_n(0)$, $f_n^{(\infty)} \equiv f_n(\infty)$, and we used the Rogers dilogarithm defined as

$$L(x) = -\frac{1}{2} \int_0^x dy \left(\frac{\ln y}{1 - y} + \frac{\ln(1 - y)}{y}\right). \tag{117}$$

This function satisfies the following relations

$$L(x) + L(1 - x) = L(1) \qquad L(x^2) = 2L(x) - 2L\left(\frac{x}{1 + x}\right) \tag{118}$$

the second of them is called the Abel duplication formula. Moreover, the following identities hold (the first is an easy consequence of the Abel formula; for a proof of the second one see, e.g., [16]),

$$\sum_{k=2}^{\nu-1} L\left(\frac{1}{k^2}\right) + 2L\left(\frac{1}{\nu}\right) = L(1) \qquad \sum_{k=1}^{n} L\left(\frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin^2\left(\frac{\pi(k+1)}{n+3}\right)}\right) = \frac{2n}{n+3} L(1) \tag{119}$$

where $n \geqslant 1$ and $\nu \geqslant 2$ are integers.

The knowledge of the entropy allows one to find the corresponding specific heat, $C_u = T\partial_T s(T)$. On the other hand, at low temperature, we have [37] $C_u = \frac{\pi c}{3v_s}T + o(T)$, where c is the effective central charge of the conformal field theory related to the $T = 0$ limit, and v_s is the speed of sound (Fermi velocity), which is $v_s = \frac{1}{2}$ for our choice of the coupling constant $J = 1/\pi$. Thus, taking into account that $L(1) = \pi^2/6$, we infer from (116) that for the alternating chain we have

$$c(\nu|\vartheta_1, \vartheta_2) = \frac{1}{L(1)} \sum_{k=1}^{\nu} \left(L\big(f_k^{(0)}\big) - L\big(f_k^{(\infty)}\big)\right) \tag{120}$$

where we introduced a new variable, $f_\nu(\lambda) \equiv 1 - f_0(\lambda)$, and used the first property in (118). Observe that the replacement of the quasi-particle corresponding to the negative parity by a formal $k = \nu$ particle makes equations (93) look uniform (since $\epsilon_\nu = -\epsilon_0$, $\eta_\nu = 1/\eta_0$). Furthermore, the formal νth particle contributes to the central charge (120) in the same way as a genuine quasi-particle. In the following analysis, this allows us to treat all the $k = 1, \ldots, \nu$ particles on equal footing.

For $T \ll 1$, the functions $\ln(1 + \eta_n(\lambda))$ vary very slowly for $|\pi\lambda| \ll -\ln T$ and for $|\pi\lambda| \gg -\ln T$. Therefore, at $H = 0, T \ll 1$ and $\lambda \to \infty$, equations (93) turn into the following system (which is the same as in the homogeneous XXZ case [14])

$$f_k^{-1} - 1 = \prod_{m=1}^{\nu} (f_m)^{-\frac{1}{2}I_{km}} \qquad k = 1, \ldots, \nu \tag{121}$$

where $f_k = f_k^{(\infty)}$, and I_{km} is the incidence (adjacency) matrix of the graph (that coincides with the Dynkin diagram of D_n with $n = \nu$)

Under the restriction $0 \leqslant f_n(\lambda) \leqslant 1$, the system (121) has a unique solution:

$$f_k^{(\infty)} = \frac{1}{(k+1)^2} \quad k \leqslant \nu - 2 \qquad f_{\nu-1}^{(\infty)} = f_\nu^{(\infty)} = \frac{1}{\nu}. \tag{122}$$

It follows from the first relation in (119) that[6]

$$\sum_{k=1}^{\nu} L\big(f_k^{(\infty)}\big) = L(1). \tag{123}$$

A remark is in order here. In the thermodynamic Bethe ansatz approach [41] to the simply laced affine Toda models (when the minimal part of the scattering matrix is considered), the corresponding effective central charge $c[\mathfrak{g}]$ is given by (120), where $f_k^{(\infty)} = 0$ and the values of $f_k^{(0)}$ satisfy the system (121) with I_{km} being the incidence matrix of the related Lie algebra $\mathfrak{g}$. From this point of view, equation (123) represents the fact that $c[D_n] = 1$, independent of n. In the case of $\mathfrak{g} = A_n$, the system (121) is solved by [14, 42]

$$f_k = \frac{\sin^2\left(\frac{\pi}{n+3}\right)}{\sin^2\left(\frac{\pi(k+1)}{n+3}\right)} \qquad k = 1, \ldots, n \tag{124}$$

---

[6] Which means, in view of (118) and the relation $f_\nu = 1 - f_0$, that the pseudo-energies at $\lambda = \infty$ do not contribute to the central charge.

which, according to the second identity in (118), implies

$$c[A_n] = \frac{2n}{n+3} \tag{125}$$

as is well known for the A_n affine Toda models.

Now we analyse equations (93) for $T \ll 1$ and $\lambda \to 0$. Consider first the case when $\vartheta_1 \leqslant \vartheta_2 < \nu - 1$. The presence of the $s(\lambda)\delta_{n\vartheta_i}$ terms in (93) leads to $\epsilon_n/T = -\infty$ for $n = \vartheta_1, \vartheta_2$. Thus, we have $f_{\vartheta_1}^{(0)} = f_{\vartheta_2}^{(0)} = 1$, which implies the following modification of the system (121)

$$\left(f_k^{(0)}\right)^{-1} - 1 = \sigma_k \prod_{m=1}^{\nu} \left(f_m^{(0)}\right)^{-\frac{1}{2}I_{km}} \qquad k = 1, \ldots, \nu \tag{126}$$

where $\sigma_k = \left(1 - \delta_{k\vartheta_1}\right)\left(1 - \delta_{k\vartheta_2}\right)$.

The system (126) corresponds to a graph which is the D_ν-type diagram with the ϑ_1 and ϑ_2 nodes removed (note that these nodes contribute $c_0 = 1$ each to the central charge). If $\vartheta_2 < \nu - 2$, the corresponding graph consists of three disconnected subgraphs—that of A_n with $n_1 = \vartheta_1 - 1$, $n_2 = \vartheta_2 - \vartheta_1 - 1$, and D_n with $n = \nu - \vartheta_2$. The contribution of the latter subgraph is $L(1)$ and it cancels in (120) against (123). Taking into account (125), we conclude that the corresponding central charge is

$$c(\nu|\vartheta_1, \vartheta_2) = c\left[A_{\vartheta_1-1}\right] + c\left[A_{\delta\vartheta-1}\right] + 2c_0 + c\left[D_{\nu-\vartheta_2}\right] - c[D_\nu] = \frac{3\vartheta_1}{\vartheta_1+2} + \frac{3\delta\vartheta}{\delta\vartheta+2} \tag{127}$$

which agrees with our numerical results (72) and with the conjectured formula [20] for the XXX alternating chain. For the special case where $(S_1, S_2) = \left(\frac{1}{2}, 1\right)$ we recover the result of [21]. Observe that (127) remains valid also for $\delta\vartheta = 0$, although the derivation should be slightly modified.

If $\vartheta_2 = \nu - 2$, the emerging D_n-type subgraph for $n = 2$ is degenerate (it consists of two disconnected nodes). But we still have $c[D_2] = 1$ because in this case equations (126) yield $f_{\nu-1}^{(0)} = f_\nu^{(0)} = \frac{1}{2}$ and the contribution of these nodes is $L(1)$, thanks to the fact that $L\left(\frac{1}{2}\right) = \frac{1}{2}L(1)$. Thus, equation (127) remains correct in this case as well.

For $\vartheta_2 = \nu - 1$, equations (93) imply that $\epsilon_0/T = +\infty$ and hence $f_{\vartheta_1}^{(0)} = f_{\vartheta_2}^{(0)} = f_\nu^{(0)} = 1$ (so these nodes contribute $3c_0 = 3$ to the central charge). Thus, we have to consider the system (126) with $\sigma_k = \left(1 - \delta_{k\vartheta_1}\right)(1 - \delta_{k,\nu-1})(1 - \delta_{k\nu})$. The corresponding graph consists (for $\delta\vartheta > 0$) of two A_n diagrams with $n_1 = \vartheta_1 - 1$ and $n_2 = \nu - \vartheta_1 - 2 = \delta\vartheta - 1$. Therefore, we conclude that the resulting central charge is given by the same expression (127). Again, if $\delta\vartheta = 0$, the derivation should be slightly modified but (127) remains valid.

4.4. Remarks on the underlying CFT

Equation (127) can be interpreted as the sum of the central charges of two $SU(2)$ WZW models at levels ϑ_1 and $\delta\vartheta$, which indicates that the structure of the effective conformal field theory is $WZW_{\vartheta_1} \otimes WZW_{\delta\vartheta}$. Furthermore, it was argued in [17] for the homogeneous XXZ chain that its underlying CFT contains composite fields formed by products of the Gaussian and the $Z(\vartheta)$-parafermionic fields in agreement with the representation $c(\nu|\vartheta) = 1 + c[A_{\vartheta-1}]$ (cf (127) for $\delta\vartheta = 0$). We may expect a similar operator content for the alternating chain; in particular, composite fields involving $Z(\vartheta_1)$ and $Z(\delta\vartheta)$ parafermions are likely to be present.

It should be remarked here that (127) gives the resulting central charge but does not reflect the fact that all ν quasi-particles in (92) contribute to its value. It is however known for the thermodynamic Bethe ansatz treatment of models with Lie algebraic symmetries, such as the

simply laced affine Toda models, that the related Lie algebraic structure is often exhibited in the quasi-particle content of characters of the related CFT [43, 44]. It is therefore natural to look for a basis in the CFT related to the (alternating) XXZ chain that would contain ν quasi-particles.

It was observed in [43] that many characters of various conformal models admit the following (so-called fermionic) representation,

$$\chi(q) = \sum_{m_1,\ldots,m_n} q^{\frac{1}{2}\mathbf{m}^t B \mathbf{m} + \mathbf{m} \cdot \mathbf{a} + a_0} \prod_{k=1}^{n} \begin{bmatrix} ((1-B)\mathbf{m} + \mathbf{u})_k \\ m_k \end{bmatrix}_q \qquad (128)$$

where B is an $n \times n$ real matrix, $\mathbf{a}$ and $\mathbf{u}$ are n-component vectors, a_0 is a constant, and the q-binomial coefficients are defined for $0 \leqslant m \leqslant n$ as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q)_n}{(q)_m (q)_{n-m}} & \text{if} \quad n < \infty \\ \dfrac{1}{(q)_m} & \text{if} \quad n = \infty \end{cases} \qquad (q)_m = \prod_{k=1}^{m} (1 - q^k). \quad (129)$$

Usually B is related to the Cartan matrix $C_{\mathfrak{g}}$ of some Lie algebra $\mathfrak{g}$, and there are typically certain restrictions on the summation over m_k. Analysis of the $q \to 1^-$ asymptotics of (128) allows one to find the corresponding central charge as a combination of Rogers dilogarithms with arguments satisfying two systems of nonlinear equations [43]. It turns out that our equations (120), (121) and (126) are exactly of this form provided that we make the following identification

$$n = \nu \qquad B = \tfrac{1}{2} C_{D_\nu} \qquad u_{\vartheta_1} = u_{\vartheta_2} = \infty \qquad u_\nu = u_{\nu-1} \qquad (130)$$

where C_{D_ν} is the Cartan matrix of the Lie algebra D_ν. Furthermore, characters of the type (128) based on $\mathfrak{g} = A_k$ arise in the context of the spinon basis of $SU(2)$ WZW models related to the homogeneous XXX spin chain [45]. This, together with the correspondence (130), makes it plausible that the spinon basis related to the (alternating) XXZ spin chain with the anisotropy $\gamma = \pi/\nu$ will correspond to (128) based on $\mathfrak{g} = D_\nu$.

Let us stress here that the correspondence (130) is based solely on the data obtained by the thermodynamic analysis of the ground state of the system. It is worth mentioning that additional information, in particular, about $\mathbf{a}$ and a_0 in (128), can be obtained from the finite size analysis of low lying excitations (see, e.g., [17, 53]) because this analysis allows one to find the anomalous dimensions of the fields present in the underlying conformal theory.

4.5. The $H = 0, T \gg 1$ limit

It is also interesting to examine the high temperature behaviour of the entropy and deduce the number of states of the model. In the high temperature limit the pseudo-energies ϵ_n become independent of λ [14]. Consequently, neglecting the free terms in (93), we obtain the same constant TBA equations as in (121). Thus, we have (cf (122))

$$1 + \eta_n = (n+1)^{2-\delta_{n,\nu-1}} \qquad n = 1, \ldots, \nu - 1. \qquad (131)$$

According to (94) and (95), the free energy density at $H = 0$ in the high temperature limit becomes

$$f = -\frac{T}{4} \sum_{n=\vartheta_1,\vartheta_2} (1 + \delta_{n,\nu-1}) \ln(1 + \eta_n). \qquad (132)$$

Hence, taking into account (131), the entropy is given by

$$S = \frac{N}{2} \sum_{n=\vartheta_1,\vartheta_2} \ln(n+1) \qquad (133)$$

which implies that (recall that $\vartheta_i = 2S_i$) the number of states for the system is

$$((2S_1 + 1)(2S_2 + 1))^{\frac{N}{2}} \tag{134}$$

as one could have expected.

5. Generalization to l spins

Consider the XXZ chain containing l different spins, $S_1, \ldots, S_l$, assigned to sites with periodicity l, i.e, the spins at the sites n and $(n + l)$ are the same. Let us outline how our analysis of the alternating chain extends to this case.

Introduce the following set of transfer matrices for $i = 1, \ldots, l$:

$$\tau^{(i)}(\lambda) = \text{tr}_a\left(R_{a,N}^{S_i S_l}(\lambda) R_{a,N-1}^{S_i S_{l-1}}(\lambda) \cdots R_{a,l+1}^{S_i S_1}(\lambda) R_{a,l}^{S_i S_l}(\lambda) \cdots R_{a,2}^{S_i S_2}(\lambda) R_{a,1}^{S_i S_1}(\lambda)\right). \tag{135}$$

They commute with each other and, extending the proof given in section 2, it can be shown that $\prod_{i=1}^{l} \tau^{(i)}(0)$ generates a shift by l lattice sites (see appendix B). This means that the Hamiltonian constructed as

$$\mathcal{H} = i\frac{J}{2l} \sum_{i=1}^{l} \partial_\lambda \ln \tau^{(i)}(\lambda)\Big|_{\lambda=0} \tag{136}$$

defines (in the gapless regime) at zero temperature a conformally invariant system. We remark that $\mathcal{H}$ is local but involves now interactions of $(l + 1)$ nearest sites.

The Bethe ansatz equations now read

$$\prod_{j=1}^{l}\left(\frac{\sinh \gamma(\lambda_\alpha + iS_j)}{\sinh \gamma(\lambda_\alpha - iS_j)}\right)^{N/l} = -\prod_{\beta=1}^{M} \frac{\sinh \gamma(\lambda_\alpha - \lambda_\beta + i)}{\sinh \gamma(\lambda_\alpha - \lambda_\beta - i)} \qquad \alpha = 1, \ldots, M. \tag{137}$$

Denote $\vartheta_i = 2S_i$. For the Bethe ansatz analysis, the order of the spins on the lattice is irrelevant, so we assume in the following that $\vartheta_1 < \vartheta_2 < \cdots < \vartheta_l < \nu$. Then, at $T = 0$, $H = 0$, the ground state in the thermodynamic limit is given by $\rho_k(\lambda) = \frac{1}{l}s(\lambda)\delta_{k\vartheta_i}$ (the Dirac sea of l different strings) with the corresponding energy

$$e_0 = -\frac{J\pi}{l^2} \sum_{i,j=1}^{l} \int_{-\infty}^{\infty} d\lambda \, Z_{\vartheta_i \vartheta_j}(\lambda) s(\lambda)$$

$$= \frac{J}{2l^2} \sum_{i,j=1}^{l} \left[\Psi_{q^2}\left(\frac{S_i + S_j}{2} + \frac{1}{2}\right) - \Psi_{q^2}\left(\frac{|S_i - S_j|}{2} + \frac{1}{2}\right)\right]. \tag{138}$$

At $T > 0$ the free energy at equilibrium is given by

$$f = e_0 - \frac{T}{l} \sum_{i=1}^{l} \left\{\int_{-\infty}^{\infty} d\lambda \, s(\lambda) \ln(1 + \eta_{\vartheta_i}(\lambda)) + \delta_{\vartheta_i, \nu-1} \int_{-\infty}^{\infty} d\lambda \, s(\lambda) \ln\left(1 + \eta_0^{-1}(\lambda)\right)\right\}. \tag{139}$$

Equations (93) for the pseudo-energies are modified only in their free terms by the replacement of $\frac{1}{2} \sum_{i=1}^{2} \delta_{k\vartheta_i}$ with $\frac{1}{l} \sum_{i=1}^{l} \delta_{k\vartheta_i}$. Observe that this change does not affect the $\lambda \to \infty$ analysis in section 4.3. For $\lambda \to 0$ we have now $f_{\vartheta_i} = 1$ for $i = 1, \ldots, l$. That is, removing l nodes, we obtain A_n diagrams with $n_1 = \vartheta_1 - 1$, $n_2 = \vartheta_2 - \vartheta_1 - 1$, etc. This yields the following central charge:

$$c(\nu|\vartheta_1, \ldots, \vartheta_l) = \frac{3\vartheta_1}{\vartheta_1 + 2} + \sum_{i=1}^{l-1} \frac{3\delta\vartheta_i}{\delta\vartheta_i + 2} \qquad \delta\vartheta_i \equiv \vartheta_{i+1} - \vartheta_i. \tag{140}$$

Note that the condition $\vartheta_i < \nu$ implies that $l \leqslant \nu - 1$. It is easy to show that, for a given ν, the maximum of (140) is attained if $l = \nu - 1$ (that is, if all the possible spins are present),

$$\max c(\nu|\{\vartheta_i\}) = c(\nu|1, 2, \ldots, \nu - 1) = \nu - 1. \tag{141}$$

On the other hand, if l is fixed, then the configuration $\vartheta_i = i$ corresponds to the minimal possible value, $c = l$.

The analysis carried out in section 4.2 extends to the case of l spins by considering the l-component counterpart of the Wiener–Hopf system (105) (with $\xi = \nu/(\nu - \vartheta_l)$ and the coefficients in front of $e^{-\pi\lambda}$ being k_i/l). Then analogues of (108) and (109) hold (with a factor $2/l$ appearing on the rhs of (109)), and we derive $k^t \hat{S}(i\pi) = \frac{l}{8\pi}(G_+(0)G_-(0))_{ll}$. Observing that $(G_+(0)G_-(0))_{ll} = \hat{A}(0)_{\vartheta_l\vartheta_l}$, we obtain $k^t \hat{S}(i\pi) = \frac{l}{4\pi}\frac{\vartheta_l}{\xi}$. Using now (139) and proceeding as in (111), we conclude that

$$f = e_0 - \frac{1}{2\pi}\frac{\nu\vartheta_l}{\nu - \vartheta_l}H^2. \tag{142}$$

Whence we infer that

$$v_s\chi = \frac{1}{2\pi}\frac{\nu\vartheta_l}{\nu - \vartheta_l} \tag{143}$$

that is, the magnetic susceptibility χ depends, for a given anisotropy, only on the maximal spin S_l.

6. Conclusion

In this paper the alternating quantum spin chain model with periodic boundary conditions and its l-spin generalization were investigated via the finite-size effects and the algebraic Bethe ansatz method. In a previous work [26] the boundary free energy for the alternating chain was computed by means of the thermodynamic Bethe ansatz method. On the other hand, the effective central charge of the open XXZ spin chain with general diagonal and non-diagonal boundaries was computed recently by means of finite-size corrections [48]. A similar computation, but for diagonal boundaries only, was also undertaken in [49]. In both cases the effective central charge clearly depends on the boundary parameters of the model. Let us also mention that the finite-size correction approach, when boundaries are present, provides somehow the same information with the TBA description in the so-called R channel [50], whereas the thermodynamic Bethe ansatz approach (see, e.g., [26, 49, 51]) can be compared with the TBA description in the L channel [49, 52]. We hope to generalize the finite-size correction approach for fused models with open boundaries.

To complete the picture, the thermodynamics of the alternating spin chain and the $RSOS(\vartheta_1, \vartheta_2; q)$ model with integrable boundaries should also be investigated. Finally, it would be of great interest to continue the thermodynamic investigation for spin chains and $RSOS$ models related to higher rank algebras. We hope to address these questions in a future work.

Acknowledgments

Appendix A. Universal XXZ *R*-matrix

The universal *R*-matrix (8) was given (in a slightly different form) in [46] without an explicit derivation. We find it instructive to provide here a proof of (8). Technically, it extends the derivation used in the case $S_1 = S_2$ [6] (see also [28, 30, 32]). We will also prove some useful related formulae.

Recall that the co-multiplication on $U_q(sl_2)$ is defined as a linear homomorphism such that

$$\Delta(S^\pm) = S^\pm \otimes q^{-S^3} + q^{S^3} \otimes S^\pm \qquad \Delta(S^3) = S^3 \otimes 1 + 1 \otimes S^3. \qquad \text{(A.1)}$$

Evaluating (6) in the representation $V_{\frac{1}{2}} \otimes V_{S_1} \otimes V_{S_2}$ (and interchanging λ with μ), we obtain

$$R_{12}^{S_1 S_2}(\lambda) L_2^{S_2}(\lambda + \mu) L_1^{S_1}(\mu) = L_1^{S_1}(\mu) L_2^{S_2}(\lambda + \mu) R_{12}^{S_1 S_2}(\lambda) \qquad \text{(A.2)}$$

which imposes the following conditions on $R^{S_1 S_2}(\lambda)$:

$$R(\lambda)\Delta' = \Delta R(\lambda) \qquad \text{(A.3)}$$

$$R(\lambda)\left(e^{2\epsilon\gamma\lambda} S^\epsilon \otimes q^{-S^3} + q^{S^3} \otimes S^\epsilon\right) = \left(e^{2\epsilon\gamma\lambda} S^\epsilon \otimes q^{S^3} + q^{-S^3} \otimes S^\epsilon\right) R(\lambda). \qquad \text{(A.4)}$$

Here (and below) $\epsilon = \pm$, and Δ' is the opposite co-multiplication, i.e., $\Delta'_q = \Delta_{1/q}$. These equations determine the universal *R*-matrix uniquely (up to a scalar factor) [6, 46]. Denote $R_\pm = (R(\pm\infty))^{-1}$. Then the asymptotics of (A.3)–(A.4) yields

$$R_\epsilon \Delta = \Delta' R_\epsilon \qquad \text{(A.5)}$$

$$R_\epsilon\left(S^\epsilon \otimes q^{S^3}\right) = \left(S^\epsilon \otimes q^{-S^3}\right) R_\epsilon \qquad R_\epsilon\left(q^{-S^3} \otimes S^{-\epsilon}\right) = \left(q^{S^3} \otimes S^{-\epsilon}\right) R_\epsilon. \qquad \text{(A.6)}$$

These equations imply that $R_\pm$ are the constant universal *R*-matrices (10). Observe that (A.5) implies that R_ϵ maps the Δ-basis in $V_{S_1} \otimes V_{S_2}$ into the Δ'-basis. That is, for the orthonormal set of eigenvectors $|j, m\rangle$, $m = j, j - 1, \ldots, -j$, of a projector $\mathcal{P}_j$ we have

$$R_\epsilon^{S_1 S_2}|j, m\rangle = \varphi_j^\epsilon|j, m\rangle' \qquad \text{(A.7)}$$

where the scalar factor φ_j^ϵ does not depend on m (its explicit form is given below).

Now we introduce $r^\pm(\lambda) = R(\lambda)R_\pm$ and then (A.3), (A.4) acquire the form

$$r^\epsilon(\lambda)\Delta = \Delta r^\epsilon(\lambda) \qquad \text{(A.8)}$$

$$r^\epsilon(\lambda)\left(e^{2\epsilon\gamma\lambda} S^\epsilon \otimes q^{S^3} + (R_\epsilon)^{-1}\left(q^{S^3} \otimes S^\epsilon\right) R_\epsilon\right) = \left(e^{2\epsilon\gamma\lambda} S^\epsilon \otimes q^{S^3} + q^{-S^3} \otimes S^\epsilon\right) r^\epsilon(\lambda). \qquad \text{(A.9)}$$

It follows from (A.8) that $r^\pm(\lambda)$ are functions of the spin operator $\mathbb{J}$, i.e.,

$$r^\pm(\lambda) = \sum_{|S_1 - S_2|}^{S_1 + S_2} r_j^\pm(\lambda)\mathcal{P}_j \qquad \text{(A.10)}$$

where $r_j^\pm(\lambda)$ are scalar functions and $\mathcal{P}_j$ is the projector onto V_j in $V_{S_1} \otimes V_{S_2}$.

Using (A.1), (A.6) and (A.7), it is straightforward to check the following relations:

$$\left[S^\epsilon \otimes q^{S^3}, \Delta(S^\epsilon)\right] = \left[q^{-S^3} \otimes S^\epsilon, \Delta(S^\epsilon)\right] = \left[(R_\epsilon)^{-1}\left(q^{S^3} \otimes S^\epsilon\right) R_\epsilon, \Delta(S^\epsilon)\right] = 0 \qquad \text{(A.11)}$$

$$\left(S^\epsilon \otimes q^{S^3} + q^{\epsilon(2+2j)} q^{-S^3} \otimes S^\epsilon\right)|j, \epsilon j\rangle = \left(1 \otimes q^{2S^3}\right)\Delta(S^\epsilon)|j, \epsilon j\rangle = 0 \qquad \text{(A.12)}$$

$$\left(\left(q^{S^3} \otimes S^\epsilon\right) R_\epsilon + q^{\epsilon(2+2j)} R_\epsilon\left(S^\epsilon \otimes q^{S^3}\right)\right)|j, \epsilon j\rangle = \left(q^{2S^3} \otimes 1\right)\Delta'(S^\epsilon)|j, \epsilon j\rangle' = 0. \qquad \text{(A.13)}$$

Equations (A.10), (A.11) show that we can apply (A.9) to a highest/lowest weight vector $|j, \epsilon j\rangle$. Doing so and taking (A.12), (A.13) into account, we obtain

$$r_{j+1}^\epsilon(\lambda)(e^{\epsilon\gamma\lambda} - q^{\epsilon(2+2j)} e^{-\epsilon\gamma\lambda}) = (e^{\epsilon\gamma\lambda} - q^{-\epsilon(2+2j)} e^{-\epsilon\gamma\lambda}) r_j^\epsilon(\lambda). \qquad \text{(A.14)}$$

Hence

$$r_{j+1}^\epsilon(\lambda) = -q^{-\epsilon(2+2j)} \frac{[j+1-i\lambda]_q}{[j+1+i\lambda]_q} r_j^\epsilon(\lambda). \tag{A.15}$$

Solving this functional equation and fixing the overall normalization, we obtain (8).

Note that (8) has the form $q^{\psi_\epsilon(\mathbb{J})}\Omega(\lambda, \mathbb{J})(R_\epsilon)^{-1}$, where the functions $\psi_\epsilon(x)$ and $\Omega(\lambda, x)$ are invariant with respect to the replacement of q by q^{-1}, and, moreover, $\Omega(\lambda, x)\Omega(-\lambda, x) = 1$, $\psi_{-\epsilon}(x) = -\psi_\epsilon(x)$. Whence, taking into account the property $(R_\epsilon)_{1/q} = (R_\epsilon)^{-1}$, we derive the first symmetry relation in (13)

$$R_{1/q}(-\lambda) = q^{-\psi_\epsilon(\mathbb{J}')}\Omega(-\lambda, \mathbb{J}')R_\epsilon = R_\epsilon q^{-\psi_\epsilon(\mathbb{J})}\big(\Omega(\lambda, \mathbb{J})\big)^{-1} = (R_q(\lambda))^{-1}. \tag{A.16}$$

Here $\mathbb{J}' = \mathbb{J}_{1/q}$ is the two-node spin operator corresponding to the opposite co-multiplication and we used that $R_\epsilon\mathbb{J} = \mathbb{J}'R_\epsilon$, as follows from (A.5).

If the algebra generators are represented in terms of matrices as in (3), then it is easy to check that $\mathbb{J}^t = \mathbb{J}$ and $(R_+)^t = (R_-)^{-1}$, where t stands for the matrix transposition. Whence we derive the second symmetry relation in (13)

$$(R(-\lambda))^t = R_{-\epsilon}q^{\psi_\epsilon(\mathbb{J})}\Omega(-\lambda, \mathbb{J}) = R_{-\epsilon}q^{-\psi_{-\epsilon}(\mathbb{J})}(\Omega(\lambda, \mathbb{J}))^{-1} = (R_q(\lambda))^{-1}. \tag{A.17}$$

It is straightforward to see from (15), (16) that the action of the ∗-operation (38) on the two-node spin operator $\mathbb{J}$ is given by $(\mathbb{J}_q)^* = \mathbb{J}_{\bar{q}}$. This, along with the property $(R_{\pm,q})^* = (R_{\mp,\bar{q}})^{-1}$, leads to the first relation in (39):

$$(R(\lambda))^* = R_{-\epsilon,\bar{q}}\Omega(-\bar{\lambda}, \mathbb{J}_{\bar{q}})\bar{q}^{-\psi_{-\epsilon}(\mathbb{J}_{\bar{q}})} = (R_{\bar{q}}(\bar{\lambda}))^{-1}. \tag{A.18}$$

The second relation in (39) follows from (22), (A.18) and the symmetries (13) and (23). For instance, for $\gamma \in \mathbb{R}$ we have

$$\begin{aligned}(\hat{R}_q(\lambda))^* &= e^{-\gamma\bar{\lambda}(1\otimes S^3)}(R_{1/q}(\bar{\lambda}))^{-1}e^{\gamma\bar{\lambda}(1\otimes S^3)}\\ &= e^{-\gamma\bar{\lambda}(1\otimes S^3)}R_q(-\bar{\lambda})e^{\gamma\bar{\lambda}(1\otimes S^3)} = \hat{R}_q(-\bar{\lambda}) = (\hat{R}_q(\bar{\lambda}))^{-1}.\end{aligned} \tag{A.19}$$

To prove the important identity (11), we employ theorem 3.1 from [47], which states (in our notation) that the $U_q(sl_2)$ Clebsch–Gordan coefficients satisfy the following relations

$$\sum_{m_1',m_2'}\big(R_+^{S_1S_2}\big)_{m_1,m_2}^{m_1',m_2'}\begin{bmatrix}S_1 & S_2 & j\\ m_1 & m_2 & m\end{bmatrix}_q = (-1)^{S_1+S_2-j}q^{\kappa(S_1)+\kappa(S_2)-\kappa(j)}\begin{bmatrix}S_2 & S_1 & j\\ m_2 & m_1 & m\end{bmatrix}_q \tag{A.20}$$

$$\begin{bmatrix}S_1 & S_2 & j\\ m_1 & m_2 & m\end{bmatrix}_q = (-1)^{S_1+S_2-j}\begin{bmatrix}S_2 & S_1 & j\\ m_2 & m_1 & m\end{bmatrix}_{q^{-1}} \tag{A.21}$$

where $\kappa(x) \equiv x(x+1)$, and $\big(R_+^{S_1S_2}\big)_{m_1,m_2}^{m_1',m_2'} = \langle m_1'|\otimes\langle m_2'|R_+^{S_1S_2}|m_1\rangle\otimes|m_2\rangle$ are matrix elements of the R-matrix in the $V_{S_1}\otimes V_{S_2}$ basis. These formulae show that for (A.7) we have

$$\varphi_j^+ = q^{\kappa(S_1)+\kappa(S_2)-\kappa(j)} \tag{A.22}$$

and also that

$$\langle m_1'|\otimes\langle m_2'|R_+^{SS}|j, m\rangle = (-1)^{2S-j}\varphi_j^+\langle m_1'|\otimes\langle m_2'|\mathbb{P}|j, m\rangle \tag{A.23}$$

which proves (11) for $\epsilon = +$. Analogous formulae for $\epsilon = -$ are obtained by observing that the map $S^+ \leftrightarrow S^-$, $S^3 \leftrightarrow -S^3$, $q \leftrightarrow q^{-1}$ is an automorphism preserving the co-multiplication (A.1). In particular, we have

$$\varphi_j^- = \big(\varphi_j^+\big)^{-1} \tag{A.24}$$

which along with (A.7) and (A.22) leads to the following relations:

$$R_-^{-1}R_+ = \sum_{j=\delta S}^{S_1+S_2} (\varphi_j^+)^2 \mathcal{P}_j \qquad R_+^{-1}R_- = \sum_{j=\delta S}^{S_1+S_2} (\varphi_j^-)^2 \mathcal{P}_j. \tag{A.25}$$

Inverting these relations, we obtain

$$\mathcal{P}_j = \prod_{\substack{l=\delta S \\ l \neq j}}^{S_1+S_2} \frac{(\varphi_l^\epsilon)^2 - R_{-\epsilon}^{-1}R_\epsilon}{(\varphi_l^\epsilon)^2 - (\varphi_j^\epsilon)^2} \qquad \varphi_j^\epsilon = q^{\epsilon(\kappa(S_1)+\kappa(S_2)-\kappa(j))}. \tag{A.26}$$

Indeed, to prove this formula, it suffices first to observe that, as seen from (A.25), the only projector which is present in all the factors of the rhs of (A.26) is $\mathcal{P}_j$, and then to check the corresponding normalization. In the simplest case, $(S_1, S_2) = (\frac{1}{2}, S)$, equation (A.26) yields

$$\mathcal{P}_{S\pm\frac{1}{2}}^{\frac{1}{2}S} = \pm\frac{1}{[2S+1]_q} \left(q^{\mp(2S+1)} - q\left(R_+^{\frac{1}{2}S}\right)^{-1}R_-^{\frac{1}{2}S}\right). \tag{A.27}$$

Consider now the universal R-matrix introduced in (22). It is a solution to equation (5) for the L-operator (20). The latter, unlike the L-operator (4), does not decompose into two Borel components. As a consequence, instead of (A.3), (A.4), we have four λ-dependent relations:

$$\hat{R}(\lambda)\left(e^{\epsilon\gamma\lambda}S^\epsilon \otimes q^{-S^3} + q^{S^3} \otimes S^\epsilon\right) = \left(e^{\epsilon\gamma\lambda}S^\epsilon \otimes q^{-S^3} + q^{S^3} \otimes S^\epsilon\right)\hat{R}(\lambda), \tag{A.28}$$

$$\hat{R}(\lambda)\left(e^{\epsilon\gamma\lambda}q^{-S^3} \otimes S^\epsilon + S^\epsilon \otimes q^{S^3}\right) = \left(e^{\epsilon\gamma\lambda}q^{S^3} \otimes S^\epsilon + S^\epsilon \otimes q^{-S^3}\right)\hat{R}(\lambda). \tag{A.29}$$

Observe that $\tilde{R}(\lambda) \equiv (\hat{R}(\lambda))^t$ solves the same set of equations, which, taking into account that the solution to (A.28), (A.29) is unique (up to a scalar factor), implies that $(\hat{R}(\lambda))^t = \zeta(\lambda)\hat{R}(\lambda)$, with $\zeta(\lambda)$ being a scalar function. In fact, $\zeta(\lambda) = \pm 1$ because $t \circ t = $ id. The possibility of $\zeta(\lambda) = -1$ is excluded by the fact that the vector $|S_1\rangle \otimes |S_2\rangle$ is an eigenvector of $\hat{R}^{S_1 S_2}(\lambda)$ with a non-vanishing eigenvalue (hence the first matrix element of $\hat{R}^{S_1 S_2}(\lambda)$ in the basis corresponding to (3) is non-vanishing). Thus, we infer that $\zeta(\lambda) = 1$, which proves the last relation in (23). The remaining symmetries in (23) are easily derived from this symmetry and the above established relations (13), for instance

$$\hat{R}(-\lambda) = e^{-\gamma\lambda(1\otimes S^3)}R(-\lambda)\,e^{\gamma\lambda(1\otimes S^3)}$$
$$= \left(e^{-\gamma\lambda(1\otimes S^3)}(R(\lambda))^t\,e^{\gamma\lambda(1\otimes S^3)}\right)^{-1} = ((\hat{R}(\lambda))^t)^{-1} = (\hat{R}(\lambda))^{-1}. \tag{A.30}$$

Finally, since $\hat{R}(0) = R(0)$, the symmetries (14) follow obviously from (13) and (23).

Appendix B. Construction of the shift operator

Let l and N be integers such that l divides N. Introduce the following invertible operators

$$Q_\pm^{i,j} = \prod_{k=i \bmod l}^N R_{k,k+j-i\pm l}^{S_i S_j}(0) \qquad \tilde{Q}^{i,j} = \prod_{k=i \bmod l}^N R_{k,k+j-i}^{S_i S_j}(0) \tag{B.1}$$

where $R_{n,m}^{S_i S_j}(\lambda)$ is the XXZ R-matrix (assigned to the sites n and m of the lattice, as in section 2). Note that in each product all the factors commute with each other iff $i \neq j \bmod l$. Along with the property (45) this implies that

$$Q_+^{i,j}Q_-^{j,i} = Q_-^{i,j}Q_+^{j,i} = 1^{\otimes N}. \tag{B.2}$$

Consider now the transfer matrices (135). It is not difficult to derive for them an analogue of equation (27),

$$\tau^{(i)}(0) = U^{(i)}Q_+^{i,i-1}Q_+^{i,i-2}\cdots Q_+^{i,1}\tilde{Q}^{i,l}\tilde{Q}^{i,l-1}\cdots\tilde{Q}^{i,i+1} \tag{B.3}$$

where $U^{(i)} = \prod_{k=i \bmod l}^{N} \mathbb{P}_{k,k+l}$ (the factors are ordered from left to right) acts nontrivially only on the nodes $n = i \bmod l$,

$$U^{(i)}\xi_n = \xi_{n+l}U^{(i)} \qquad \text{if} \quad n \equiv i \bmod l. \tag{B.4}$$

Observe that $U = U^{(1)}U^{(2)}\cdots U^{(l)}$ is the lattice shift operator (generating a shift by l sites).

Note that $\tilde{Q}^{i,j}U^{(j)} = U^{(j)}Q_-^{i,j}$. Therefore (taking all $U^{(j)}$ to the left) we have

$$
\begin{aligned}
\tau^{(1)}(0)\tau^{(2)}(0)\cdots\tau^{(l)}(0) = {}& U\,Q_-^{1,l}Q_-^{1,l-1}\cdots Q_-^{1,3}Q_-^{1,2}\\
& \times Q_+^{2,1}Q_-^{2,l}\cdots Q_-^{2,4}Q_-^{2,3}\\
& \qquad\qquad \cdots\\
& \times Q_+^{l-1,l-2}Q_+^{l-1,l-3}\cdots Q_+^{l-1,1}Q_-^{l-1,l}\\
& \times Q_+^{l,l-1}Q_+^{l,l-2}\cdots Q_+^{l,2}Q_+^{l,1}.
\end{aligned}
\tag{B.5}
$$

Note that here we have $Q_-^{i,j}$ if $i < j$ and $Q_+^{i,j}$ if $i > j$. Let us show that all Q_- can be led to meet their Q_+ counterparts and thus be cancelled by relation (B.2). As the first step, the rightmost Q_- in each row, i.e, $Q_-^{i,i+1}$, $i = 1, \ldots, l-1$, is cancelled against the leftmost Q_+ in the next row. Observe that this eliminates obstacles for moving then the second from the right Q_- through the entire next row (obviously, $Q_\pm^{i,j}$ commutes with $Q_\pm^{i',j'}$ if i, j, i', j' are all distinct). Thus, at the second step, $Q_-^{i,i+2}$ for $i = 1, \ldots, l-2$ are cancelled against $Q_+^{i+2,i}$ (which, after the first step, became the leftmost factors in their rows). Proceeding this way, we eliminate at the kth step $l - k$ pairs $\left(Q_-^{i,i+k}, Q_+^{i+k,i}\right)$ and also make possible the next step (that is, $Q_-^{i,i+k+1}$ will be able to move through the next k rows). Consequently, after the $(l-1)$th step we will have all Q eliminated, and so we conclude that $\tau^{(1)}(0)\cdots\tau^{(l)}(0) = U$.

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